Problems

3.1 [1] Show that the radial velocity along a Kepler orbit is

$$\dot{r} = \frac{GMe}{L}\sin(\psi - \psi_0), \tag{3.324}$$

where L is the angular momentum. By considering this expression in the limit $r \to \infty$ show that the eccentricity e of an unbound Kepler orbit is related to its speed at infinity by

$$e^2 = 1 + \left(\frac{Lv_\infty}{GM}\right)^2. \tag{3.325}$$

3.2 [1] Show that for a Kepler orbit the eccentric anomaly η and the true anomaly $\psi - \psi_0$ are related by

$$\cos(\psi - \psi_0) = \frac{\cos\eta - e}{1 - e\cos\eta} \quad ; \quad \sin(\psi - \psi_0) = \sqrt{1 - e^2}\frac{\sin\eta}{1 - e\cos\eta}. \tag{3.326}$$

3.3 [1] Show that the energy of a circular orbit in the isochrone potential (2.47) is $E = -GM/(2a)$, where $a = \sqrt{b^2 + r^2}$. Let the angular momentum of this orbit be $L_c(E)$. Show that

$$L_c = \sqrt{GMb}\left(x^{-1/2} - x^{1/2}\right), \qquad \text{where} \qquad x \equiv -\frac{2Eb}{GM}. \tag{3.327}$$

3.4 [1] Prove that if a homogeneous sphere of a pressureless fluid with density ρ is released from rest, it will collapse to a point in time $t_{\rm ff} = \frac{1}{4}\sqrt{3\pi/(2G\rho)}$. The time $t_{\rm ff}$ is called the **free-fall time** of a system of density ρ.

3.5 [3] Generalize the timing argument in Box 3.1 to a universe with non-zero vacuum-energy density. Evaluate the required mass of the Local Group for a universe of age $t_0 = 13.7\,\text{Gyr}$ with (a) $\Omega_{\Lambda 0} = 0$; (b) $\Omega_{\Lambda 0} = 0.76$, $h_7 = 1.05$. Hints: the energy density in radiation can be neglected. The solution requires evaluation of an integral similar to (1.62).

3.6 [1] A star orbiting in a spherical potential suffers an arbitrary instantaneous velocity change while it is at pericenter. Show that the pericenter distance of the ensuing orbit cannot be larger than the initial pericenter distance.

3.7 [2] In a spherically symmetric system, the apocenter and pericenter distances are given by the roots of equation (3.14). Show that if $E < 0$ and the potential $\Phi(r)$ is generated by a non-negative density distribution, this equation has either no root, a repeated root, or two roots (Contopoulos 1954). Thus there is at most one apocenter and pericenter for a given energy and angular momentum. Hint: take the second derivative of $E - \Phi$ with respect to $u = 1/r$ and use Poisson's equation.

3.8 [1] Prove that circular orbits in a given potential are unstable if the angular momentum per unit mass on a circular orbit decreases outward. Hint: evaluate the epicycle frequency.

3.9 [2] Compute the time-averaged moments of the radius, $\langle r^n \rangle$, in a Kepler orbit of semi-major axis a and eccentricity e, for $n = 1, 2$ and $n = -1, -2, -3$.

3.10 [2] $\Delta\psi$ denotes the increment in azimuthal angle during one complete radial cycle of an orbit.

(a) Show that in the potential (3.57)

$$\Delta\psi = \frac{2\pi L}{\sqrt{-2Er_{\rm a}r_{\rm p}}}, \tag{3.328}$$

where $r_{\rm a}$ and $r_{\rm p}$ are the apo- and pericentric radii of an orbit of energy E and angular momentum L. Hint: by contour integration one can show that for $A > 1$, $\int_{-\pi/2}^{\pi/2} {\rm d}\theta/(A + \sin\theta) = \pi/\sqrt{A^2 - 1}$.

(b) Prove in the epicycle approximation that along orbits in a potential with circular frequency $\Omega(R)$,

$$\Delta\psi = 2\pi\left(4 + \frac{\mathrm{d}\ln\Omega^2}{\mathrm{d}\ln R}\right)^{-1/2}. \tag{3.329}$$

(c) Show that the exact expression (3.328) reduces for orbits of small eccentricity to (3.329).

3.11 [1] For what spherically symmetric potential is a possible trajectory $r = ae^{b\psi}$?

3.12 [2] Prove that the mean-square velocity is on a bound orbit in a spherical potential $\Phi(r)$ is

$$\langle v^2 \rangle = \left\langle r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right\rangle, \tag{3.330}$$

where $\langle\cdot\rangle$ denotes a time average.

3.13 [2] Let $\mathbf{r}(s)$ be a plane curve depending on the parameter s. Then the **curvature** is

$$K = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \tag{3.331}$$

where $\mathbf{r}' \equiv \mathrm{d}\mathbf{r}/\mathrm{d}s$. The local radius of curvature is K^{-1}. Prove that the curvature of an orbit with energy E and angular momentum L in the spherical potential $\Phi(r)$ is

$$K = \frac{L\,\mathrm{d}\Phi/\mathrm{d}r}{2^{3/2}r[E - \Phi(r)]^{3/2}}. \tag{3.332}$$

Hence prove that no orbit in any spherical mass distribution can have an inflection point (in contrast to the cover illustration of Goldstein, Safko, & Poole 2002).

3.14 [1] Show that in a spherical potential the vertical and circular frequencies ν and Ω (eqs. 3.79) are equal.

3.15 [1] Prove that at any point in an axisymmetric system at which the local density is negligible, the epicycle, vertical, and circular frequencies κ, ν, and Ω (eqs. 3.79) are related by $\kappa^2 + \nu^2 = 2\Omega^2$.

3.16 [1] Using the epicycle approximation, prove that the azimuthal angle $\Delta\psi$ between successive pericenters lies in the range $\pi \le \Delta\psi \le 2\pi$ in the gravitational field arising from any spherical mass distribution in which the density decreases outwards.

3.17 [3] The goal of this problem is to prove the results of Problem 3.16 without using the epicycle approximation (Contopoulos 1954).

(a) Using the notation of §3.1, show that

$$E - \Phi - \frac{L^2}{2r^2} = (u_1 - u)(u - u_2)\left\{\tfrac{1}{2}L^2 + \Phi[u, u_1, u_2]\right\}, \tag{3.333}$$

where $u_1 = 1/r_1$ and $u_2 = 1/r_2$ are the reciprocals of the pericenter and apocenter distances of the orbit respectively, $u = 1/r$, and

$$\Phi[u, u_1, u_2] = \frac{1}{u_1 - u_2}\left[\frac{\Phi(u_1) - \Phi(u)}{u_1 - u} - \frac{\Phi(u) - \Phi(u_2)}{u - u_2}\right]. \tag{3.334}$$

This expression is a second-order divided difference of the potential Φ regarded as a function of u, and a variant of the mean-value theorem of calculus shows that $\Phi[u, u_1, u_2] = \frac{1}{2}\Phi''(\bar{u})$ where $\bar{u}$ is some value of u in the interval (u_1, u_2). Then use the hint in Problem 3.7 and equation (3.18b) to deduce that $\Delta\psi \le 2\pi$ when the potential Φ is generated by a non-negative, spherically symmetric density distribution.

(b) A lower bound on $\Delta\psi$ can be obtained from working in a similar manner with the function

$$\chi(\omega) = \frac{2\omega\Phi}{L}, \quad \text{where} \quad \omega \equiv \frac{L}{r^2}. \tag{3.335}$$

Show that

$$\frac{2\omega E}{L} - \chi(\omega) - \omega^2 = (\omega_1 - \omega)(\omega - \omega_2)\left\{1 + \chi[\omega, \omega_1, \omega_2]\right\}, \tag{3.336}$$

where $\omega_1 = L/r_1^2$, $\omega_2 = L/r_2^2$ and $\chi[\omega, \omega_1, \omega_2]$ is a second-order divided difference of $\chi(\omega)$. Now deduce that $\Delta\psi \geq \pi$ for any potential in which the circular frequency $\Omega(r)$ decreases outwards.

3.18 [1] Let $\Phi(R, z)$ be the Galactic potential. At the solar location, $(R, z) = (R_0, 0)$, prove that

$$\frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho_0 + 2(A^2 - B^2), \tag{3.337}$$

where ρ_0 is the density in the solar neighborhood and A and B are the Oort constants. Hint: use equation (2.73).

3.19 [3] Consider an attractive power-law potential, $\Phi(r) = Cr^\alpha$, where $-1 \leq \alpha \leq 2$ and $C > 0$ for $\alpha > 0$, $C < 0$ for $\alpha < 0$. Prove that the ratio of radial and azimuthal periods is

$$\frac{T_r}{T_\psi} = \begin{cases} 1/\sqrt{2+\alpha} & \text{for nearly circular orbits} \\ \begin{cases} 1/2, & \text{for } \alpha > 0 \\ 1/(2+\alpha), & \text{for } \alpha < 0 \end{cases} & \text{for nearly radial orbits.} \end{cases} \tag{3.338}$$

What do these results imply for harmonic and Kepler potentials?
Hint: depending on the sign of α use a different approximation in the radical for v_r. For $b > 0$, $\int_1^\infty \mathrm{d}x/(x\sqrt{x^b - 1}) = \pi/b$ (see Touma & Tremaine 1997).

3.20 [1] Show that in spherical polar coordinates the Lagrangian for motion in the potential $\Phi(\mathbf{x})$ is

$$\mathcal{L} = \tfrac{1}{2}[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] - \Phi(\mathbf{x}). \tag{3.339}$$

Hence show that the momenta p_θ and p_ϕ are related to the the magnitude and z-component of the angular-momentum vector $\mathbf{L}$ by

$$p_\phi = L_z \quad ; \quad p_\theta^2 = L^2 - \frac{L_z^2}{\sin^2\theta}. \tag{3.340}$$

3.21 [3] Plot a $(y, \dot{y})$, $(x = 0, \dot{x} > 0)$ surface of section for motion in the potential Φ_L of equation (3.103) when $q = 0.9$ and $E = -0.337$. Qualitatively relate the structure of this surface of section to the structure of the $(x, \dot{x})$ surface of section shown in Figure 3.9.

3.22 [3] Sketch the structure of the $(x, \dot{x})$, $(y = 0, \dot{y} > 0)$ surface of section for motion at energy E in a Kepler potential when (a) the (x, y) coordinates are inertial, and (b) the coordinates rotate at 0.75 times the circular frequency Ω at the energy E. Hint: see Binney, Gerhard, & Hut (1985).

3.23 [3] The Earth is flattened at the poles by its spin. Consequently orbits in its potential do not conserve total angular momentum. Many satellites are launched in inclined, nearly circular orbits only a few hundred kilometers above the Earth's surface, and their orbits must remain nearly circular, or they will enter the atmosphere and be destroyed. Why do the orbits remain nearly circular?

3.24 [2] Let $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ be unit vectors in an inertial coordinate system centered on the Sun, with $\hat{\mathbf{e}}_1$ pointing away from the Galactic center (towards $\ell = 180°$, $b = 0$) and $\hat{\mathbf{e}}_2$ pointing towards $\ell = 270°$, $b = 0°$. The mean velocity field $\mathbf{v}(\mathbf{x})$ relative to the Local Standard of Rest can be expanded in a Taylor series,

$$v_i = \sum_{j=1}^{2} H_{ij}x_j + \mathrm{O}(x^2). \tag{3.341}$$

(a) Assuming that the Galaxy is stationary and axisymmetric, evaluate the matrix $\mathbf{H}$ in terms of the Oort constants A and B.

(b) What is the matrix $\mathbf{H}$ in a rotating frame, that is, if $\hat{\mathbf{e}}_1$ continues to point to the center of the Galaxy as the Sun orbits around it?

(c) In a homogeneous, isotropic universe, there is an analogous 3×3 matrix $\mathbf{H}$ that describes the relative velocity $\mathbf{v}$ between two fundamental observers separated by $\mathbf{x}$. Evaluate this matrix in terms of the Hubble constant.

3.25 [3] Consider two point masses m_1 and $m_2 > m_1$ that travel in a circular orbit about their center of mass under their mutual attraction. (a) Show that the Lagrange point L_4 of this system forms an equilateral triangle with the two masses. (b) Show that motion near L_4 is stable if $m_1/(m_1 + m_2) < 0.03852$. (c) Are the Lagrange points L_1, L_2, L_3 stable? See Valtonen & Karttunen (2006).

3.26 [2] Show that the leapfrog integrator (3.166a) is second-order accurate, in the sense that the errors in $\mathbf{q}$ and $\mathbf{p}$ after a timestep h are $\mathrm{O}(h^3)$.

3.27 [2] Forest & Ruth (1990) have devised a symplectic, time-reversible, fourth-order integrator of timestep h by taking three successive drift-kick-drift leapfrog steps of length ah, bh, and ah where $2a + b = 1$. Find a and b. Hint: a and b need not both be positive.

3.28 [2] Confirm the formulae for the Adams–Bashforth, Adams–Moulton, and Hermite integrators in equations (3.169), (3.170), and (3.171), and derive the next higher order integrator of each type. You may find it helpful to use computer algebra.

3.29 [1] Prove that the fictitious time τ in Burdet–Heggie regularization is related to the eccentric anomaly η by $\tau = (T_r/2\pi a)\eta + \mathit{constant}$, if the motion is bound ($E_2 < 0$) and the external field $\mathbf{g} = 0$.

3.30 [1] We wish to integrate numerically the motions of N particles with positions $\mathbf{x}_i$, velocities $\mathbf{v}_i$, and masses m_i. The particles interact only by gravitational forces (the gravitational N-body problem). We are considering using several possible integrators: modified Euler, leapfrog, or fourth-order Runge–Kutta. Which of these will conserve the total momentum $\sum_{i=1}^N m_i\mathbf{v}_i$? Which will conserve the total angular momentum $\sum_{i=1}^N m_i\mathbf{x}_i \times \mathbf{v}_i$? Assume that all particles are advanced with the same timestep, and that forces are calculated exactly. You may solve the problem either analytically or numerically.

3.31 [2] Show that the generating function of the canonical transformation from angle-action variables (θ_i, J_i) to the variables (q_i, p_i) discussed in Box 3.4 is

$$S(q, J) = \mp\tfrac{1}{2}q\sqrt{2J - q^2} \pm J\cos^{-1}\left(\frac{q}{\sqrt{2J}}\right). \tag{3.342}$$

3.32 [1] Let $\epsilon(R)$ and $\ell(R)$ be the specific energy and angular momentum of a circular orbit of radius R in the equatorial plane of an axisymmetric potential.

(a) Prove that

$$\frac{\mathrm{d}\ell}{\mathrm{d}R} = \frac{R\kappa^2}{2\Omega} \quad ; \quad \frac{\mathrm{d}\epsilon}{\mathrm{d}R} = \tfrac{1}{2}R\kappa^2, \tag{3.343}$$

where Ω and κ are the circular and epicycle frequencies.

(b) The energy of a circular orbit as a function of angular momentum is $\epsilon(\ell)$. Show that $\mathrm{d}\epsilon/\mathrm{d}\ell = \Omega$ in two ways, first from the results of part (a) and then using angle-action variables.

3.33 [2] The angle variables θ_i conjugate to the actions J_i can be implicitly defined by the coupled differential equations $\mathrm{d}w_\alpha/\mathrm{d}\theta_i = [w_\alpha, J_i]$, where w_α is any ordinary phase-space