Problems

7.1 [1] Consider a system in which the interparticle potential energy has the form $\Phi_{\alpha\beta} = -C|\mathbf{x}_\alpha - \mathbf{x}_\beta|^{-p}$, where p and C are positive constants.

(a) Show that the scalar virial theorem has the form

$$2K + pW = 0, \tag{7.203}$$

where K is the kinetic energy and W is the potential energy.

(b) For what values of p does the system have negative heat capacity, in the sense of equation (7.51)?

7.2 [1] Prove that a system of N self-gravitating point masses with positive energy must disrupt, in the sense that at least one star must escape. Hint: use the virial theorem, and prove that the moment of inertia must increase without limit.

7.3 [2] A simple model for a galactic disk consists of N infinite, parallel sheets, each having surface density σ. Let z be the coordinate perpendicular to the sheets and label the position of the ith sheet by z_i.

(a) Show that the equation of motion is

$$\ddot{z}_i = 2\pi G\sigma(N_{+i} - N_{-i}), \tag{7.204}$$

where N_{+i} is the number of sheets with $z > z_i$ and N_{-i} is the number with $z < z_i$.

(b) Show that the Hamiltonian of the system can be written as

$$H = \frac{1}{2\sigma}\sum_{i=1}^{N} p_i^2 + \pi G\sigma^2 \sum_{\substack{i,j=1 \\ i\neq j}}^{N} |z_i - z_j|, \tag{7.205}$$

where p_i is the momentum conjugate to z_i.

(c) If the first and second terms in the Hamiltonian are identified with the kinetic energy K and potential energy W per unit area, show that the virial theorem has the form

$$2K = W \quad \text{or} \quad E \equiv K + W = 3K. \tag{7.206}$$

7.4 [1] Consider a stellar system composed of two types of stars, with density distributions $\rho_1(\mathbf{x})$ and $\rho_2(\mathbf{x})$ and corresponding potentials $\Phi_1(\mathbf{x})$ and $\Phi_2(\mathbf{x})$. Show that in a steady state, the scalar virial theorem for component 2 may be written in the form

$$2K_2 + W_2 - \int \mathrm{d}^3\mathbf{x}\, \rho_2(\mathbf{x})\mathbf{x}\cdot\boldsymbol{\nabla}\Phi_1(\mathbf{x}) = 0, \tag{7.207}$$

where K_2 is the kinetic energy of component 2, and W_2 is the potential energy due to the mutual interaction of the stars of component 2. Hint: use Problem 4.38.

7.5 [2] The **Klimontovich distribution function** for an N-body system of identical point masses m is

$$f(\mathbf{x},\mathbf{v},t) = \sum_{\alpha=1}^{N} \delta[\mathbf{x} - \mathbf{x}_\alpha(t)]\delta[\mathbf{v} - \mathbf{v}_\alpha(t)], \tag{7.208}$$

where δ denotes the three-dimensional delta function (Appendix C.1). The functions $\mathbf{x}_\alpha(t)$, $\mathbf{v}_\alpha(t)$ describe the trajectories of the N bodies, which satisfy the equations

$$\dot{\mathbf{x}}_\alpha = \mathbf{v}_\alpha \quad ; \quad \dot{\mathbf{v}}_\alpha = Gm\sum_{\beta\neq\alpha} \frac{\mathbf{x}_\beta - \mathbf{x}_\alpha}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}, \tag{7.209}$$

(a) Prove that the Klimontovich DF is an exact solution of the collisionless Boltzmann equation (4.11) for the one-body Hamiltonian

$$H(\mathbf{x},\mathbf{v},t)=\tfrac{1}{2}v^2-\sum_{\alpha=1}^{N}\frac{Gm}{|\mathbf{x}_\alpha(t)-\mathbf{x}|}. \tag{7.210}$$

The Klimontovich DF offers an exact formal description of the N-body stellar system, and provides an alternative to the BBGKY hierarchy for the systematic derivation of kinetic equations to describe stellar systems (Nishikawa & Wakatani 2000).

(b) We have argued that the collisionless Boltzmann equation cannot account for relaxation due to encounters between individual stars. How is this consistent with our conclusion that the Klimontovich DF is an exact solution of the N-body equations of motion?

7.6 [2] The object of this problem is to determine the behavior of the curve in Figure 7.1 in the limit as the central concentration $\mathcal{R}\to\infty$.

(a) The density of an isothermal sphere satisfies equation (4.107a),

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}\left(\widetilde{r}^2\frac{\mathrm{d}\ln\widetilde{\rho}}{\mathrm{d}\widetilde{r}}\right)=-9\widetilde{r}^2\widetilde{\rho}. \tag{7.211}$$

As the dimensionless radius $\widetilde{r}\to\infty$, the solutions of equation (7.211) approach the singular isothermal sphere $\widetilde{\rho}_{\mathrm{S}}(\widetilde{r})=2/(9\widetilde{r}^2)$. To determine the asymptotic behavior more accurately, define new variables u and $z(u)$ by $u=1/\widetilde{r}$ and $\widetilde{\rho}\equiv\widetilde{\rho}_{\mathrm{S}}(\widetilde{r})\exp(z)$. Show that equation (7.211) becomes

$$u^2\frac{\mathrm{d}^2z}{\mathrm{d}u^2}+2(\mathrm{e}^z-1)=0. \tag{7.212}$$

(b) By linearizing equation (7.212) for small z, show that the asymptotic behavior of the density $\widetilde{\rho}$ is described by the equation (Chandrasekhar 1939)

$$\widetilde{\rho}(\widetilde{r})\simeq\widetilde{\rho}_{\mathrm{S}}(\widetilde{r})\Big[1+\frac{A}{\widetilde{r}^{1/2}}\cos\left(\tfrac{1}{2}\sqrt{7}\ln\widetilde{r}+\phi\right)\Big], \tag{7.213}$$

where A and ϕ are constants determined by the boundary conditions at small radii. Thus, at large $\widetilde{r}$, the density of the isothermal sphere *oscillates* around the singular solution, with fractional amplitude decreasing as $\widetilde{r}^{-1/2}$.

(c) Now consider an isothermal gas enclosed in a spherical box of radius r_{b}, with inverse temperature β (cf. eq. 7.55). As $\widetilde{r}_{\mathrm{b}}\to\infty$, show that the mass M of gas can be written in the form

$$x\equiv\frac{r_{\mathrm{b}}}{GMm\beta}\simeq\tfrac{1}{2}-\frac{A}{8\widetilde{r}_{\mathrm{b}}^{1/2}}\Big[\cos\left(\tfrac{1}{2}\sqrt{7}\ln\widetilde{r}_{\mathrm{b}}+\phi\right)+\sqrt{7}\sin\left(\tfrac{1}{2}\sqrt{7}\ln\widetilde{r}_{\mathrm{b}}+\phi\right)\Big]. \tag{7.214}$$

Hence, argue that when the central concentration $\mathcal{R}=1/\widetilde{\rho}(\widetilde{r}_{\mathrm{b}})$ is large, the curve in Figure 7.1 becomes vertical at successive values of $\mathcal{R}$ that are in the ratio $\exp(4\pi/\sqrt{7})=115.54$.

7.7 [1] A particle undergoes a one-dimensional random walk, defined as follows. If the particle is at position x, then during any short time interval Δt the mean-square change in position is $\langle(\Delta x)^2\rangle=D(x)\Delta t$, while the mean change is $\langle\Delta x\rangle=0$. Let $p(x,t)\mathrm{d}x$ be the probability that the particle is found in the interval $(x,x+\mathrm{d}x)$ at time t. What is the partial differential equation governing $p(x,t)$?

7.8 [2] Consider a D-dimensional stellar system containing N identical stars that interact by inverse-square forces ($D=2$, flat disk; $D=3$, sphere, etc.). Show that the relaxation time and the crossing time in such a system are related by

$$t_{\mathrm{relax}}\approx\begin{cases}t_{\mathrm{cross}}, & D=2,\\ N\ln N\,t_{\mathrm{cross}}, & D=3,\\ Nt_{\mathrm{cross}}, & D>3.\end{cases} \tag{7.215}$$

7.9 [2] A subject mass M is embedded in an infinite homogeneous sea of field stars, with mass $m \ll M$ and isotropic DF $f(v)$. Using the Fokker–Planck equation and the diffusion coefficients (7.88), show that when the subject mass is in thermal equilibrium with the field stars its DF is Maxwellian, with velocity dispersion

$$\sigma_M^2 = \frac{m}{M}\frac{\int_0^\infty \mathrm{d}v\, v f(v)}{f(0)}. \tag{7.216}$$

Show that when $f(v)$ is Maxwellian, this condition reduces to the requirement of energy equipartition between the subject mass and the field stars.

7.10 [2] A subject mass M is embedded in an infinite homogeneous sea of field stars with number density $n(m)\mathrm{d}m$ in the mass range $(m, m+\mathrm{d}m)$. The field stars of all masses have a Maxwellian DF with dispersion σ. Thus the field stars are not in thermal equilibrium, which would require that σ^2 is inversely proportional to m. Let us assume, however, that the subject star *is* in thermal equilibrium with the field stars (this could occur, for example, if $M \gg m$).

(a) Show that the dispersion of the subject star is

$$\sigma_M^2 = \sigma^2 \frac{\int \mathrm{d}m\, m^2 n(m)}{M \int \mathrm{d}m\, m n(m)}. \tag{7.217}$$

(b) Show that the DF of the subject star is Maxwellian.

7.11 [2] The diffusion coefficients for a Maxwellian field star distribution depend on the functions $\mathrm{G}(X)$ and $\mathrm{erf}(X) - \mathrm{G}(X)$, where $\mathrm{G}(X)$ is defined by equation (7.93) and $X = v/(\sqrt{2}\sigma)$. Show that

$$\lim_{X\to 0} \frac{\mathrm{erf}(X)}{X} = \frac{2}{\sqrt{\pi}} \quad ; \quad \lim_{X\to 0} \frac{\mathrm{G}(X)}{X} = \frac{2}{3\sqrt{\pi}}. \tag{7.218}$$

Thus show that as the velocity of the subject star $v \to 0$, the diffusion coefficients of equation (7.92) satisfy $D[(\Delta v_\parallel)^2] = \frac{1}{2}D[(\Delta \mathbf{v}_\perp)^2]$. Explain physically why this must be so.

7.12 [2] Suppose that a spherical cluster evolves self-similarly as a result of relaxation. In this case its evolution can be described by two functions $M(t)$ and $R(t)$, the mass and characteristic radius as functions of time. Since the evolution is driven by relaxation, we expect that

$$\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{C_M}{t_{\mathrm{relax}}} \quad ; \quad \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{C_R}{t_{\mathrm{relax}}} \tag{7.219}$$

where C_M and C_R are constants of order unity. Neglecting changes in the Coulomb logarithm, the relaxation time $t_{\mathrm{relax}} \propto R^{3/2}M^{1/2}$ (eq. 7.108).

(a) If the evolution is dominated by evaporation, we expect that $M(t)$ declines with time while the cluster energy $E \propto GM^2/R$ remains constant, since evaporating stars leave with nearly zero energy. In this case show that

$$M(t) \propto \tau^{2/7} \quad ; \quad R(t) \propto \tau^{4/7}, \tag{7.220}$$

where τ is the time remaining until the cluster disappears.

(b) After core collapse, the evolution of a cluster is dominated by the energy input from binary stars at the cluster center, so the cluster energy E grows but the mass M remains approximately constant. In this case show that

$$R(t) \propto \tau^{2/3}, \tag{7.221}$$

where τ is the time elapsed since core collapse (Goodman 1984).

7.13 [1] Using equation (7.173) for the evaporation time of soft binaries, estimate the maximum semi-major axis of a primordial soft binary that could survive for 10 Gyr in the solar neighborhood. Assume that the DF in the solar neighborhood is isotropic and Maxwellian, with RMS velocity $50\,\mathrm{km\,s^{-1}}$, that all stars have mass $1\,\mathcal{M}_\odot$, and that the stellar density is $\rho = 0.04\,\mathcal{M}_\odot\,\mathrm{pc}^{-3}$ (from Tables 1.1 and 1.2).

7.14 [3] A population of very hard binaries, each with total mass m_b and internal energy $\widetilde{E}$, is embedded in a distribution of field stars of mass m. The velocity distributions of the field stars and of the centers of mass of the binaries are Maxwellian, with dispersions σ and σ_b respectively. Show that the disruption rate of the binaries contains an exponential factor

$$\exp\left[-\frac{(m+m_\mathrm{b})|\widetilde{E}|}{mm_\mathrm{b}(\sigma^2+\sigma_\mathrm{b}^2)}\right], \tag{7.222}$$

which reduces to the exponential factor in equation (7.174) when $m_\mathrm{b} = 2m$, $\sigma_\mathrm{b}^2 = \frac{1}{2}\sigma^2$, and $|\widetilde{E}| \gg m\sigma^2$.

7.15 [1] What is the closest approach that a star is likely to have made to the Sun during its lifetime of 4.5 Gyr, assuming that the Sun's environment has always been similar to the present solar neighborhood? Use the same parameters for the solar neighborhood as in Problem 7.13.

7.16 [1] A tidal-capture binary is formed as a result of a close encounter of two stars of equal mass m. The minimum separation during the encounter is d_min, and the orbital energy dissipated in the encounter is $\Delta E \ll Gm^2/d_\mathrm{min}$. Once the binary has formed, more energy is dissipated in each successive orbit, until eventually the binary orbit is circularized. If the spin angular momentum of the stars is negligible compared to the orbital angular momentum, show that the radius of the final circular orbit is $2d_\mathrm{min}$.

7.17 [2] This problem investigates the distribution of nearest neighbors of stars and the forces from them. Consider an infinite, homogeneous system of stars of mass m and number density n, and assume that the DF is separable, that is, that the two-body correlation function is negligible (§7.2.4).

(a) Show that the probability that the nearest neighbor of a star lies within distance r is

$$1 - \mathrm{e}^{-4\pi n r^3/3}. \tag{7.223}$$

(b) Show that the probability that the force per unit mass exerted on a star by its nearest neighbor lies in the range $(F, F + \mathrm{d}F)$ is

$$\mathrm{d}p_F = \frac{\mathrm{d}F}{F_0} W(F/F_0), \quad \text{where} \quad F_0 = Gmn^{2/3} \tag{7.224}$$

and

$$W(\xi) = \frac{2\pi}{\xi^{5/2}} \exp\left(-\frac{4\pi}{3\xi^{3/2}}\right). \tag{7.225}$$

(c) The probability distribution of the total force per unit mass exerted on a star by *all* of its neighbors can be shown to be given by equation (7.224) with

$$W(\xi) = \frac{2}{\pi\xi}\int_0^\infty \mathrm{d}x\, x \sin x\, \mathrm{e}^{-(sx/\xi)^{3/2}}, \quad \text{where} \quad s = 2\pi\left(\tfrac{4}{15}\right)^{2/3}. \tag{7.226}$$

This is the **Holtsmark distribution** (Chandrasekhar 1943b). Using numerical or analytic methods, show that the expressions (7.225) and (7.226) agree for large ξ.

(d) The total force is the sum of a large number of random variables (the forces from individual neighbor stars). Why then is the probability distribution (7.226) not Gaussian, as implied by the central limit theorem (Appendix B.10)? See Feller (1971) for a thorough discussion of the relation between the Holtsmark and Gaussian distributions.

7.18 [2] A black hole of mass $M_\bullet$ is embedded in the center of an infinite, homogeneous, three-dimensional sea of test particles. Far from the hole, the test particles have a Maxwellian velocity distribution (7.91) with number density n_0 and velocity dispersion σ. Show that the density distribution of test particles that are not bound to the hole is

$$n(r) = n_0 \left\{ 2\sqrt{\frac{r_\mathrm{F}}{\pi r}} + \mathrm{e}^{r_\mathrm{F}/r}\left[1 - \mathrm{erf}\left(\sqrt{\frac{r_\mathrm{F}}{r}}\right)\right]\right\}, \tag{7.227}$$

where the error function $\mathrm{erf}(x)$ is defined in Appendix C.3, and $r_\mathrm{F} = GM_\bullet/\sigma^2$. Show that close to the hole, $r \ll r_\mathrm{F}$, $n(r) \propto r^{-1/2}$. Thus there is a weak density cusp around the hole, similar in structure to the cusps seen in luminous elliptical galaxies (Nakano & Makino 1999).

7.19 [3] In §7.5.9b we derived the steady-state density distribution of stars around a central black hole, in the case where the relaxation time is shorter than the age of the system. Consider the following alternate derivation. Assume that the density near the hole is a power law, $n(r) \propto r^{-s}$. The mean-square velocity at any radius should be of order $\langle v^2 \rangle \simeq GM_\bullet/r$, so the local relaxation time is $t_\mathrm{relax} \approx \langle v^2\rangle^{3/2}/(G^2m^2n) \propto r^{s-3/2}$. Relaxation among the $N(r)$ cusp stars interior to r can lead to a flow of stars through the shell at radius r that is of order $N(r)/t_\mathrm{relax} \sim n(r)r^3/t_\mathrm{relax} \sim r^{-2s+9/2}$. In a steady state, this flow must be independent of radius, so $s = \frac{9}{4}$. This differs from the (correct) result $s = \frac{7}{4}$ derived in §7.5.9b. What is wrong with the argument presented here?