Problems

4.1 [1] Show that in a frame that rotates with constant angular velocity $\mathbf{\Omega}$, with $\Phi_{\rm eff} \equiv \Phi - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2$, the collisionless Boltzmann equation can be written

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - [2(\mathbf{\Omega} \times \mathbf{v}) + \nabla \Phi_{\rm eff}] \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{4.284}$$

Hint: see §3.3.2.

4.2 [2] Consider an infinite homogeneous system of collisionless zero-mass test particles in D-dimensional space. The particles have an isotropic velocity distribution $f(v)$. Initially the particles are subject to no forces. At $t = 0$ a gravitational potential well suddenly appears in a finite region of the space. Show that as $t \to \infty$, the density of unbound particles traveling through the well is smaller than the asymptotic density if $D = 1$, larger if $D = 3$, and unchanged if $D = 2$.

4.3 [1] A spherical mass distribution is immersed in a sea of collisionless test particles, which arrive with velocities $\mathbf{v} = (v, 0, 0)$ from the negative x-direction and are scattered by the gravitational field from the mass. Does the DF of the test particles satisfy the Jeans theorem? If so, write down the DF as a function of the integrals of motion; if not, explain why the Jeans theorem fails.

4.4 [1] Prove that the density of a spherical, ergodic, self-consistent stellar system must decrease outward. Hint: in the integral for ρ make Φ the integration variable.

4.5 [3] A spherical galaxy with a constant mass-to-light ratio and an ergodic DF has an $R^{1/4}$ surface-brightness profile (eq. 1.17 with $m = 4$). The luminosity-weighted line-of-sight velocity dispersion within the effective radius $R_{\rm e}$ is $\sigma_{\rm e}$, that is

$$\sigma_{\rm e}^2 = \frac{\int_0^{R_{\rm e}} \mathrm{d}R\, R I(R) \sigma_\parallel^2(R)}{\int_0^{R_{\rm e}} \mathrm{d}R\, R I(R)}. \tag{4.285}$$

Show that the total mass of the galaxy can be written in the form

$$M = k \frac{\sigma_{\rm e}^2 R_{\rm e}}{G}, \tag{4.286}$$

and evaluate k numerically.

4.6 [2] The DF of a spherical system is proportional to $L^\gamma f(\mathcal{E})$. Show that at all radii the anisotropy parameter is $\beta = -\frac{1}{2}\gamma$.

4.7 [1] Show that a Hernquist model with constant anisotropy $\beta = \frac{1}{2}$ has

$$N(E) = \frac{3a}{GM} \widetilde{\mathcal{E}}^2 \left(\frac{1}{\widetilde{\mathcal{E}}} - 1 \right)^2, \tag{4.287}$$

where $\widetilde{\mathcal{E}} = \mathcal{E}a/(GM)$ and M and a are the mass and scale radius of the Hernquist model.

4.8 [2] Consider a spherical system with DF $f(\mathcal{E}, L)$. Let $N(\mathcal{E}, L)\mathrm{d}\mathcal{E}\mathrm{d}L$ be the fraction of stars with $\mathcal{E}$ and L in the ranges $(\mathcal{E}, \mathcal{E} + d\mathcal{E})$ and $(L, L + \mathrm{d}L)$.

(a) Show that

$$N(\mathcal{E}, L) = 8\pi^2 L f(\mathcal{E}, L) T_r(\mathcal{E}, L), \tag{4.288}$$

where T_r is the radial period defined by equation (3.17).

(b) A spherical system of test particles with ergodic DF surrounds a point mass. Show that the fraction of particles with eccentricities in the range $(e, e + \mathrm{d}e)$ is $2e\,\mathrm{d}e$.

4.9 [1] Consider the DF

$$f(\mathcal{E},L)=\begin{cases}F\delta_+(L^2)(\mathcal{E}-\mathcal{E}_0)^{-1/2} & (\mathcal{E}>\mathcal{E}_0)\\ 0 & (\mathcal{E}\le\mathcal{E}_0),\end{cases} \tag{4.289a}$$

where F and $\mathcal{E}_0$ are constants. Here $\delta_+(x)\equiv\delta(x-\epsilon)$, where ϵ is a small positive number; thus $\delta_+(x)=0$ for $x\neq 0$ and $\int_0^\infty \mathrm{d}x\,\delta_+(x)=1$ (cf. Appendix C.1). Show that this DF self-consistently generates a model with density

$$\rho(r)=\begin{cases}Cr^{-2} & (r<r_0)\\ 0 & (r\ge r_0),\end{cases} \tag{4.289b}$$

where C is a constant and $\Psi(r_0)=\mathcal{E}_0$ (Fridman & Polyachenko 1984).

4.10 [2] Consider a spherical system in which at every radius the star density in velocity space is constant on ellipsoidal figures of rotation. Show that the DF has the form $f=f(Q)$, where $Q(\mathcal{E},L)$ is defined by equation (4.73).

4.11 [2] Prove that the following DF generates a stellar system in which the density distribution is that of a homogeneous sphere of density ρ and radius a:

$$f(E,L)=\frac{9}{16\pi^4 G\rho a^5}\frac{1}{\sqrt{L^2/a^2+\frac{4}{3}\pi G\rho a^2-2E}}\qquad (L^2<\tfrac{4}{3}\pi G\rho a^4). \tag{4.290}$$

Here it is understood that $f=0$ when the argument of the square root is not positive, the DF is normalized so that $\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}\,f=1$, the potential $\Phi=0$ at $r=0$, and the system is isolated (Polyachenko & Shukhman 1973).

4.12 [2] Show that the Osipkov–Merritt model that self-consistently generates the Jaffe model has DF

$$\begin{aligned}f(Q)=\frac{1}{2\pi^3(GMa)^{3/2}}\Bigg[&F_-\left(\sqrt{2\widetilde{Q}}\right)-\sqrt{2}F_-\left(\sqrt{\widetilde{Q}}\right)\\ &-\sqrt{2}\left(1+\frac{a^2}{2r_\mathrm{a}^2}\right)F_+\left(\sqrt{\widetilde{Q}}\right)+\left(1+\frac{a^2}{r_\mathrm{a}^2}\right)F_+\left(\sqrt{2\widetilde{Q}}\right)\Bigg],\end{aligned} \tag{4.291}$$

where $\widetilde{Q}=aQ/(GM)$.

4.13 [2] Show that when the DF of a spherical system depends only on the function $Q(\mathbf{x},\mathbf{v})$ defined by equation (4.73), the ratio of the mean-square tangential and radial speeds is

$$\overline{v_\mathrm{t}^2}\Big/\overline{v_r^2}=\frac{2}{1+(r/r_\mathrm{a})^2}. \tag{4.292}$$

4.14 [1] A self-consistent stellar system has an ergodic DF and a power-law density profile $\rho=\rho_0(r_0/r)^\alpha$ with $1<\alpha<3$. Show that the velocity dispersion is given by

$$\overline{v_r^2}(r)=\frac{2\pi G\rho_0 r_0^\alpha r^{2-\alpha}}{(3-\alpha)(\alpha-1)}\qquad(\alpha\neq 2). \tag{4.293}$$

What does this formula become in the case $\alpha=2$ of the singular isothermal sphere?

4.15 [1] Solve the Lane–Emden equation (4.91b) for the case $n=1$, to show that

$$\psi=\begin{cases}\dfrac{\sin\sqrt{3}s}{\sqrt{3}s} & (s<\pi/\sqrt{3})\\[2ex] \dfrac{\pi}{\sqrt{3}s}-1 & (s\ge\pi/\sqrt{3}).\end{cases} \tag{4.294}$$

Show that the model's total mass is $M=\frac{1}{2}\Psi_0 G^{-3/2}\sqrt{\pi/c_1}$, where c_1 is defined by equation (4.85b).

4.16 [1] Consider a stellar system having a DF of the form $f \propto \mathcal{E}^{-1/2}$ for $\mathcal{E} > 0$ and zero otherwise. The density and potential are related by $\rho = c_1\Psi$ for $\Psi > 0$, and zero otherwise (eq. 4.85). Prove that Poisson's equation is satisfied if the density of the system has the form

$$\rho(x, y, z) = A\cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{\pi y}{2L}\right)\cos\left(\frac{\pi z}{2L}\right), \tag{4.295}$$

for $|x|, |y|, |z| \leq L$ and zero otherwise, where $L^2 = 3\pi/(16Gc_1)$. Does this mean we can construct a cubical galaxy?

4.17 [2] An extension to the polytropes described in §4.3.3a is obtained by considering spherical stellar systems with the DF

$$f(E) = FE^{-n-3/2} \quad (E \geq 0); \tag{4.296}$$

here f is defined so that $\int \mathrm{d}^3\mathbf{v}\, f = \rho$ and the potential is defined so that $\Phi = 0$ at the center of the system.

(a) Show that the density satisfies

$$\rho = d_n\Phi^{-n} \quad (\Phi > 0), \tag{4.297}$$

and evaluate d_n. What values of n are allowed?

(b) Show that the dimensionless radius $s \equiv r/b$ and potential $\phi \equiv \Phi/\Phi_0$ satisfy the equation

$$\frac{1}{s^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right) = 3\phi^{-n}, \tag{4.298}$$

where Φ_0 is arbitrary and $b \equiv (\frac{4}{3}\pi G\Phi_0^{-n-1}d_n)^{-1/2}$. What is the stellar system described by these equations in the limit $n \to \infty$?

(c) Show that these equations admit power-law solutions of the form $\rho \propto r^{-\alpha}$ for $n > 0$, with $\alpha = 2n/(1+n)$ so $0 < \alpha \leq 2$.

(d) In Problem 4.14 we found self-consistent power-law stellar systems with $1 < \alpha < 3$. Why does the current approach not find the systems with $2 < \alpha < 3$? Why does the approach in Problem 4.14 not find the solutions with $0 < \alpha \leq 1$?

4.18 [1] For a Maxwellian distribution of velocities with one-dimensional dispersion σ, show that: (a) the mean speed is $\overline{v} = (8/\pi)^{1/2}\sigma$; (b) the mean-square speed is $\overline{v^2} = 3\sigma^2$; (c) the mean-square of one component of velocity is $\overline{v_x^2} = \sigma^2$; (d) the mean-square relative speed of any two particles is $\overline{v_{\rm rel}^2} = 6\sigma^2$; (e) the fraction of particles with $v^2 > 4\overline{v^2}$ is 0.00738.

4.19 [1] At large radii, the density in a Michie model (eq. 4.117) is dominated by stars with $\mathcal{E}/\sigma^2 \ll 1$. In this case, show that the density can be written in the form

$$\rho \propto \int_0^{\sqrt{2\Psi}} \mathrm{d}v_{\rm t}\, v_{\rm t} \exp\left(-\frac{r^2v_{\rm t}^2}{2r_{\rm a}^2\sigma^2}\right)(2\Psi - v_{\rm t}^2)^{3/2}. \tag{4.299}$$

Hence show that as the tidal radius $r_{\rm t}$ is approached, ρ tends to zero as

$$\rho \propto \Psi^{5/2} \propto (r_{\rm t} - r)^{5/2}. \tag{4.300}$$

4.20 [1] Show that in a Kepler potential, the Schwarzschild DF (4.153) is equivalent to the **Rayleigh distribution** of eccentricities and inclinations,

$$\mathrm{d}n \propto ei\exp\left(-\frac{e^2}{e_0^2} - \frac{i^2}{i_0^2}\right)\mathrm{d}e\,\mathrm{d}i. \tag{4.301}$$

What is the relation of e_0^2 to σ_R^2 and i_0^2 to σ_z^2?

4.21 [2] We may study the vertical structure of a thin axisymmetric disk by neglecting all radial derivatives and assuming that all quantities vary only in the coordinate z normal to the disk. Thus we adopt the form $f = f(E_z)$ for the DF, where $E_z \equiv \frac{1}{2}v_z^2 + \Phi(z)$. Show that for an isothermal disk in which $f = \rho_0(2\pi\sigma_z^2)^{-1/2}\exp(-E_z/\sigma_z^2)$, the approximate form (2.74) of Poisson's equation may be written

$$2\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} = \mathrm{e}^{-\phi}, \quad \text{where} \quad \phi \equiv \frac{\Phi}{\sigma_z^2}, \quad \zeta \equiv \frac{z}{z_0}, \quad \text{and} \quad z_0 \equiv \frac{\sigma_z}{\sqrt{8\pi G\rho_0}}. \tag{4.302a}$$

By solving this equation subject to the boundary conditions $\phi(0) = \phi'(0) = 0$, show that the density in the disk is given by (Spitzer 1942)

$$\rho(z) = \rho_0\,\mathrm{sech}^2(\tfrac{1}{2}z/z_0). \tag{4.302b}$$

Show further that the surface density of the disk is

$$\Sigma = \frac{\sigma_z^2}{2\pi G z_0} = 4\rho_0 z_0. \tag{4.302c}$$

4.22 [3] Determine the density $\rho(z)$ of an isothermal distribution of stars with dispersion σ_z in a disk that also contains a razor-thin layer of gas in the midplane, with surface density Σ_{g}. Hint: generalize the results of Problem 4.21.

4.23 [2] Using the one-dimensional approximation of Problem 4.21, write a numerical procedure that finds the fraction $F(z)$ of stars that reach a maximum height above the midplane that exceeds z. Show that $F(z_0) = 0.808$. Find $F(2z_0)$.

4.24 [3] Every star in a spherical system loses mass slowly and isotropically. If the initial DF is $f_0(\mathcal{E}, L)$, show that after every star has been reduced to a fraction p of its original mass, the DF will be

$$f_p(\mathcal{E}, L) = f_0(p^{-2}\mathcal{E}, L). \tag{4.303}$$

How has the density profile of the system changed? Hint: see Richstone & Potter (1982).

4.25 [2] A spherical stellar system has surface brightness $I(R)$, line-of-sight velocity dispersion $\sigma_\parallel(R)$, and constant mass-to-light ratio. From these functions, together with the Jeans equations, can we deduce uniquely the luminosity density $j(r)$, the radial dispersion profile, $\sigma_r^2(r)$, and the anisotropy parameter $\beta(r)$? Hints: (i) consider a change in the dispersions of the form $\Delta\sigma_r^2 = 2\epsilon/(jr^3)$, $\Delta\sigma_\theta^2 = -\frac{1}{2}\Delta\sigma_r^2$; (ii) the answer may depend on whether $I(R)$ and $\sigma_\parallel(R)$ are known over a finite range of radii or at *all* radii (see Dejonghe & Merritt 1992).

4.26 [3] A finite, spherical stellar system of test particles is confined by the potential $\Phi(r) = v_{\mathrm{c}}^2 \ln r$.

(a) If the DF is ergodic, prove that the number of stars with line-of-sight velocity in the interval $(v_\parallel, v_\parallel + \mathrm{d}v_\parallel)$ is $n(v_\parallel)\,\mathrm{d}v_\parallel$, where

$$n(v_\parallel) \propto \exp\left(-3v_\parallel^2/2v_{\mathrm{c}}^2\right). \tag{4.304}$$

(b) If the DF is radial, that is, if all of the test particles have zero angular momentum, prove that the distribution of line-of-sight velocities is given by

$$n(v_\parallel) \propto E_1\left(v_\parallel^2/2v_{\mathrm{c}}^2\right), \tag{4.305}$$

where E_1 is the exponential integral.

4.27 [1] Show that a self-gravitating isothermal stellar system with velocity dispersion σ, cylindrical symmetry, and non-singular, non-zero density ρ_0 at $R = 0$ has the density distribution

$$\rho(R, \phi, z) = \rho_0\left(1 + \frac{\pi G\rho_0 R^2}{2\sigma^2}\right)^{-2}. \tag{4.306}$$

4.28 [1] In a spherical stellar system with mass profile $M(r)$, a stellar population with number density $n(r)$ has anisotropy parameter (4.61) of the form $\beta(r) = r^2/(r_a^2 + r^2)$, where r_a is a constant. Show that

$$\overline{v_r^2}(r) = \frac{G\int_r^\infty \mathrm{d}r\,[(r_a/r')^2 + 1]n(r')M(r')}{(r_a^2 + r^2)n(r)}. \tag{4.307}$$

4.29 [2] Let us write the general nth-order velocity moment in spherical coordinates in the form $\overline{v_\theta^{n-j} v_\phi^{j-k} v_r^k}$. Prove that when $f = f(H)$ (i) the moment vanishes if any of j, k, or n is odd; (ii) if j, k, and n are all even, then

$$\overline{v_\theta^{n-j} v_\phi^{j-k} v_r^k} = \overline{v_r^n}\,\frac{\left(\frac{k-1}{2}\right)!\left(\frac{n-j-1}{2}\right)!\left(\frac{j-k-1}{2}\right)!}{\pi\left(\frac{n-1}{2}\right)!}. \tag{4.308}$$

4.30 [2] When the DF of an axisymmetric system has the form $f(H, L_z)$, show that the nth-order velocity moments are related by

$$\overline{v_R^{n-j+2} v_z^{j-k-2} v_\phi^k} = \frac{n-j+1}{j-k-1}\,\overline{v_R^{n-j} v_z^{j-k} v_\phi^k}, \tag{4.309}$$

where j, k, and n are all even. Moments that contain odd powers of either v_R or v_z vanish. Hence the independent non-vanishing nth order moments may be taken to be $\overline{v_\phi^n}$, $\overline{v_\phi^{n-2} v_R^2}$, $\overline{v_\phi^{n-4} v_R^4}$, ...

4.31 [1] Show that in a stellar-dynamical polytrope $\overline{v_r^2} \propto \Psi$. Show that for a Plummer model the coefficient of proportionality is $\frac{1}{6}$.

4.32 [2] The velocity dispersion in some axisymmetric stellar system is isotropic and a function $\sigma(\rho)$ of the density alone. Show that the mean azimuthal velocity must be a function $\overline{v}_\phi(R)$ of the cylindrical radius R only. Is this configuration physically plausible?

4.33 [2] A static, spherically symmetric stellar system with ergodic DF is confined by a spherical vessel of radius r_b. Show that $2K + W = 4\pi r_b^3 p$, where K and W are the system's kinetic and potential energies, and $p = \rho\overline{v_r^2}$ is the pressure exerted by the system on the vessel's walls.

4.34 [1] Suppose the principal axes of the velocity ellipsoid near the Sun are always parallel to the unit vectors of spherical coordinates. Then show that for $|z|/R$ small, $\overline{v_R v_z} \simeq (\overline{v_r^2} - \overline{v_\theta^2})(z/R)$.

4.35 [1] A stationary stellar system of negligible mass and finite extent is confined by the potential $\Phi(r) = v_c^2 \ln r + \mathit{constant}$.

(a) Prove that the mean-square velocity is $\langle v^2 \rangle = v_c^2$, independent of the shape, radial profile, or other properties of the stellar system. Hint: as in the derivation of the virial theorem, consider the behavior of $\mathrm{d}^2 I/\mathrm{d}t^2$, where in this case $I = r^2$.

(b) The singular isothermal sphere has the same potential (eq. 4.104), but in this system the mean-square velocity is $\langle v^2 \rangle = 3\sigma^2 = \frac{3}{2}v_c^2$. How is this consistent with the result of part (a)?

4.36 [2] The energy per unit mass of a star in a stationary stellar system can be written $\epsilon = \frac{1}{2}v^2 + \Phi$, where v is the speed of the star and $\Phi(\mathbf{x})$ is the gravitational potential. Prove that the total energy of the stellar system is

$$\widetilde{E} = \tfrac{1}{3}M\langle\epsilon\rangle, \tag{4.310}$$

where M is the total mass of the system and $\langle\cdot\rangle$ denotes a mass-weighted average over the stars.

4.37 [1] Show that the part of equation (4.239) that is antisymmetric in j and k is equivalent to the law of conservation of angular momentum.

4.38 [1] Show that in the presence of an externally generated gravitational potential $\Phi_{\rm ext}$, the right side of equation (4.247) acquires an extra term:

$$V_{jk} \equiv -\tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}\left(x_k\frac{\partial\Phi_{\rm ext}}{\partial x_j}+x_j\frac{\partial\Phi_{\rm ext}}{\partial x_k}\right)\rho. \tag{4.311}$$

4.39 [2] In this problem we use the tensor virial theorem to connect the shape of a bar, its pattern speed, and the extent to which there is less motion parallel to the axis of figure rotation than in the perpendicular directions. Let the z axis coincide with a principal axis of the tensor $\mathbf{I}$ (eq. 4.243), and suppose that the density distribution is stationary in a frame that rotates about this axis with angular frequency Ω. Show that at an instant when $I_{xy}=0$, the left side of equation (4.247) is Ω^2 times the diagonal tensor with components $(I_{yy}-I_{xx})$, $(I_{xx}-I_{yy})$, and 0 along the diagonal. Hence show that

$$\Omega^2 = -\frac{(W_{xx}-W_{yy})+2(T_{xx}-T_{yy})+(\Pi_{xx}-\Pi_{yy})}{2(I_{xx}-I_{yy})}, \tag{4.312a}$$

and if $T_{zz}=0$,

$$\frac{v_0^2}{\sigma_0^2} = (1-\delta)\frac{W_{xx}+W_{yy}}{W_{zz}} - 2, \tag{4.312b}$$

where $v_0^2 \equiv 2(T_{xx}+T_{yy})/M$, $\sigma_0^2 \equiv (\Pi_{xx}+\Pi_{yy})/2M$ and $(1-\delta)(\Pi_{xx}+\Pi_{yy}) \equiv 2\Pi_{zz}$.

4.40 [1] Suppose that the Oort limit has been determined as described in §4.9.3 from observations of stars whose distances have been systematically overestimated by a factor λ. By what factor is the derived local mass density $\rho(0)$ in error, if the kinematics are derived from (a) radial velocities; (b) proper motions?

4.41 [2] Consider a hypothetical disk galaxy in which all the mass is contained in a central point mass. The disk density is negligible; more precisely, the disk consists of a population of stars of zero mass with RMS z-velocity σ_z that is independent of z. At radius R, the number density of these stars as a function of z is $\nu(z)=\nu(0)\exp(-z^2/2z_0^2)$, where $z_0 \ll R$ is a constant. (a) What is the relation between σ_z and z_0? (b) What does equation (4.273) predict for the local mass density if these stars are used as tracers? Why is the wrong answer obtained?

4.42 [2] Consider a time-independent, self-gravitating collisionless stellar system with slab symmetry, that is, a system in which the density ρ depends only on a single coordinate z. Prove that the system must be symmetric, that is, with a suitable choice of the coordinate origin $\rho(z)=\rho(-z)$.

4.43 [3] A natural model DF for a razor-thin axisymmetric disk is given by equation (4.147),

$$f(H,L_z) = S(L_z)\exp[-\Delta/\sigma_R^2(L_z)], \tag{4.313}$$

where $\Delta = H - E_{\rm c}(L_z)$ and $E_{\rm c}(L_z)$ is the energy of a circular orbit with angular momentum L_z. For a disk with surface density $\Sigma(R) \propto \exp(-R/R_{\rm d})$, in a potential with a constant circular speed v_0, we may take $\sigma_R^2(L_z) \propto \exp[-L_z/(R_{\rm d}v_0)]$ and $S(L_z) \propto \Sigma(L_z/v_0)/\sigma_R^2(L_z) \propto$ *constant* (see eq. 4.156 and the following discussion). With these assumptions, the DF in the solar neighborhood, at radius $R=R_0$, depends on the dimensionless parameters $\xi \equiv R_0/R_{\rm d}$ and $b \equiv \sigma_R(R_0v_0)/v_0$, which is $\ll 1$ in a cool disk. Show that in the solar neighborhood

$$\begin{aligned}\overline{v}_\phi &= v_0 + (\tfrac{1}{4}-\xi)\frac{w}{v_0} + (\tfrac{1}{32}+\tfrac{5}{12}\xi+\tfrac{3}{2}\xi^2-\tfrac{9}{8}\xi^3)\frac{w^2}{v_0^3} + \mathrm{O}(w^3),\\ \overline{(v_\phi-\overline{v}_\phi)^2} &= \tfrac{1}{2}w - (\tfrac{1}{8}+\xi-\tfrac{5}{4}\xi^2)\frac{w^2}{v_0^2} + \mathrm{O}(w^3),\end{aligned} \tag{4.314}$$

where $w \equiv \overline{v_R^2}$. Relate the O($w$) terms to epicycle theory (§3.2.3) and Stromberg's asymmetric drift equation (§4.8.2a). The O(w^2) terms provide convenient analytic estimates for the errors incurred in using the epicyle and Stromberg approximations. Hint: use computer algebra.

4.44 [2] (a) By taking a suitable moment of the collisionless Boltzmann equation, show that in a steady-state axisymmetric galaxy

$$\frac{\partial(\nu\overline{v_R^2 v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z v_\phi})}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi^3} - \overline{v}_\phi R\frac{\partial\Phi}{\partial R}\right) + \frac{2\nu}{R}\overline{v_R^2 v_\phi} = 0. \tag{4.315}$$

(b) Given that the system is symmetric in z, and that all odd moments of $v_\phi - \overline{v}_\phi$ vanish, so $0 = \overline{v_R^2(v_\phi - \overline{v}_\phi)}$ and $0 = \overline{(v_\phi - \overline{v}_\phi)^3}$, etc., show that at $z = 0$

$$\overline{v_R^2}\left(\frac{\partial\overline{v}_\phi}{\partial R} + \frac{\overline{v}_\phi}{R}\right) - \frac{2}{R}\overline{v}_\phi\overline{(v_\phi - \overline{v}_\phi)^2} = 0. \tag{4.316}$$

Hence using equation (4.222a), show that (cf. eq. 3.100)

$$\frac{\sigma_\phi^2}{\sigma_R^2} \equiv \frac{\overline{(v_\phi - \overline{v}_\phi)^2}}{\overline{v_R^2}} \simeq \frac{-B}{A - B}, \tag{4.317}$$

where A and B are the Oort constants (eq. 3.83). What is the most questionable assumption made in this derivation? Explain why violations of equation (4.317) increase with σ_R, and compare this result to the results of Problem 4.43.

4.45 [3] A rotating axisymmetric stellar system has a star density in velocity space that is constant on ellipsoids, that is, the DF at a given position depends on velocity $\mathbf{v}$ only through the combination $Q = \sum_{i,j} s_{ij}(v_i - \overline{v}_i)(v_j - \overline{v}_j)$, where $\overline{\mathbf{v}} = \overline{v}_\phi\hat{\mathbf{e}}_\phi$ is the mean azimuthal velocity.[22]

(a) If the DF depends only on $\mathcal{E}$ and L_z, prove that the rotation curve must have the form

$$\overline{v}_\phi(R, z) = \frac{R}{a + bR^2}, \tag{4.318}$$

where a and b are constants.

(b) If the velocity distribution is isotropic (constant on spheres in velocity space), so $s_{ij} = s\delta_{ij}$, prove that the system rotates at constant angular velocity, that is, the constant b in equation (4.318) is zero.

(c) Prove that result (b) holds for any stationary DF, even if it depends on a third integral.

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[22] Systems of this kind were a major early focus of research in stellar dynamics, because Schwarzschild's observation that the velocity distribution was ellipsoidal in the solar neighborhood (§4.4.3) led theorists to explore the **ellipsoidal hypothesis** that the distribution was ellipsoidal at all points in the Galaxy. See Chandrasekhar (1942).