proportional to L). Thus the spatial resolution of the force field that one obtains from $\widetilde{\Phi}$ is very large near the center of the system, and this makes the multipole expansion ideal for simulations of highly centrally concentrated systems such as elliptical galaxies. It can also be adapted to handle colliding galaxies.

Van Albada (1982), Villumsen (1982), and McGlynn (1982, 1984) have all performed N-body simulations using multipole expansions. McGlynn (1982) discusses some refinements that help to reduce two-body relaxation in simulations of this type and minimize problems that can arise from an unfortunate choice of center. This approach to finding the forces acting in N-body simulations can be extended to schemes using expansions in other coordinate systems such as cylindrical and bispherical coordinates (Villumsen 1984; Piran & Villumsen 1987).

Problems

2-1. [1] Astronauts orbiting an unexplored planet find that (i) the surface of the planet is precisely spherical; and (ii) the potential exterior to the planetary surface is $\Phi = -GM/r$ exactly, that is, there are no non-zero multipole moments of higher order than the monopole. Can they conclude from these observations that the mass distribution in the interior of the planet is spherically symmetric? If not, give a simple example of a nonspherical mass distribution that would reproduce the observations.

2-2. [1] Show that the gravitational potential energy of a spherical system can be written

$$W = -\frac{G}{2}\int_0^\infty \frac{M^2(r)dr}{r^2}, \tag{2P-1}$$

where $M(r)$ is the mass interior to radius r.

2-3. [1] Show that the potential generated by the spherical density distribution (Jaffe 1983)

$$\rho(r) = \left(\frac{M}{4\pi r_J^3}\right)\frac{r_J^4}{r^2(r+r_J)^2} \quad \text{is} \quad \Phi(r) = \frac{GM}{r_J}\ln\left(\frac{r}{r+r_J}\right), \tag{2P-2}$$

where M and r_J are constants. Verify that the total mass of the system is M. Show that the circular speed is approximately constant at $r \ll r_J$, and falls off as $v_c \propto r^{-\frac{1}{2}}$ at $r \gg r_J$.

2-4. [1] Prove that the Chandrasekhar potential energy tensor for any spherical body has the form $W_{jk} = \frac{1}{3}W\delta_{jk}$, where W is the potential energy.

2-5. [1] Defining **prolate spheroidal coordinates** (u, v) by $R = a \sinh u \sin v$, $z = a \cosh u \cos v$, where a is a constant, show that $R^2 + (a + |z|)^2 = a^2(\cosh u + |\cos v|)^2$. Hence show that the potential (2-49a) of Kuzmin's disk can be written

$$\Phi_K(u, v) = -\left(\frac{GM}{a}\right) \frac{\cosh u - |\cos v|}{\sinh^2 u + \sin^2 v}. \tag{2P-3}$$

In §3.5 we show that this potential is an example of a Stäckel potential, in which orbits admit an extra isolating integral.

2-6. [2] Consider an axisymmetric body whose density distribution is $\rho(R, z)$ and total mass is $M = \int \rho(R, z) d^3\mathbf{r}$. Assume that the body has finite extent $[\rho(R, z) = 0$ for $r^2 = R^2 + z^2 > r^2_{\text{max}}]$ and is symmetric about its equator, that is, $\rho(R, -z) = \rho(R, z)$.

(a) Show that at distances large compared to r_{max}, the potential arising from this body can be written in the form

$$\Phi(R, z) \simeq -\frac{GM}{r} - \frac{G}{4} \frac{(R^2 - 2z^2)}{r^5} \int \rho(R', z')(R'^2 - 2z'^2) d^3\mathbf{r}', \tag{2P-4}$$

where the fractional error is of order $(r_{\text{max}}/r)^2$ smaller than the second term.

(b) Show that at large distances from an exponential disk with surface density $\Sigma(R) = \Sigma_0 \exp(-R/R_d)$, the potential has the form

$$\Phi(R, z) \simeq -\frac{GM}{r} \left[1 + \frac{3R_d^2(R^2 - 2z^2)}{2r^4}\right], \tag{2P-5}$$

where M is the mass of the disk.

2-7. [2] Prove that the external potentials and force fields of any two confocal spheroids of uniform density and equal mass are everywhere the same.

2-8. [2] Use equation (2-99) to show that a prolate body with density $\rho = \rho_0(1 + R^2/a_1^2 + z^2/a_3^2)^{-2}$, where $a_3 > a_1$, generates the potential

$$\Phi(u, v) = -\pi G a_1^2 a_3 \rho_0 \int_0^\infty \frac{\sqrt{a_3^2 + \tau}\, d\tau}{(\tau + a_3^2 + \lambda)(\tau + a_3^2 + \mu)}, \tag{2P-6}$$

where (u, v) are oblate spheroidal coordinates defined by equation (2-58) with $\Delta^2 = a_3^2 - a_1^2$, and we have written $\lambda \equiv \Delta^2 \sinh^2 u$, $\mu \equiv -\Delta^2 \cos^2 v$. Decompose the integral in (2P-6) into partial fractions to show (without evaluating the integrals) that Φ is of the special Stäckel form discussed in §3.5. Finally, show that

$$\Phi(u, v) = -\left(\frac{2\pi G a_1^2 a_3 \rho_0}{\Delta^2}\right) \frac{f(\Delta \sinh u) - f(i\Delta \cos v)}{\sinh^2 u + \cos^2 v}, \tag{2P-7a}$$

where

$$f(z) \equiv z \arctan(z/a_3). \tag{2P-7b}$$

[Hint: To ensure convergence of the integrals, you may wish to add $(\tau + a_3^2)^{-\frac{1}{2}}$ to one of the integrands and subtract it from the other.] De Zeeuw (1985) calls the body with this potential the **perfect prolate spheroid**, because it is the only prolate axisymmetric density distribution of constant ellipticity whose potential is of the Stäckel form.

2-9. [2] Show that the analog to equation (2-159) that relates the potential $\Phi(R, \phi, z)$ to the surface density $\Sigma(R, \phi)$ for a non-axisymmetric disk is

$$\Phi(R,\phi,z) = -G \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^\infty dk e^{-k|z|} J_m(kR) \int_0^\infty dR' R' J_m(kR') \times \int_0^{2\pi} \Sigma(R',\phi') e^{-im\phi'} d\phi', \tag{2P-8}$$

where $J_m(u)$ is the cylindrical Bessel function of order m.

2-10. [2] The purpose of this problem is to reproduce an elegant method due to Schwarzschild (1954) of evaluating the potential energy W of a finite spherical system that has a constant mass-to-light ratio Υ.

(a) Show that the surface brightness $I(R)$ and luminosity density $j(r)$ are related by the formula

$$I(R) = 2\int_R^\infty \frac{j(r) r\, dr}{\sqrt{r^2 - R^2}}. \tag{2P-9}$$

(b) Invert equation (2P-9) using Abel's formula [see eq. (1B-59)] to obtain

$$j(r) = -\frac{1}{\pi}\int_r^\infty \frac{dI(R)}{dR}\frac{dR}{\sqrt{R^2 - r^2}}. \tag{2P-10}$$

(c) The **strip brightness** $S(x)$ is defined so that $S(x)dx$ is the total luminosity in a strip of width dx that passes a distance x from the projected center of the system. Show that

$$S(x) = 2\int_x^\infty \frac{I(R) R\, dR}{\sqrt{R^2 - x^2}}. \tag{2P-11}$$

(d) Show that the strip brightness and luminosity density are related by

$$j(x) = -\frac{1}{2\pi x}\frac{dS(x)}{dx}. \tag{2P-12}$$

(e) Prove that the mass interior to radius r is given by

$$M(r) = -2\Upsilon \int_0^r \frac{dS}{dx} x\, dx. \tag{2P-13}$$

(f) Using equation (2-130) for W show that

$$W = -2G\Upsilon^2 \int_0^\infty S^2(x)\, dx. \tag{2P-14}$$

This result was used by Schwarzschild and Bernstein (1955) in one of the first measurements of the mass-to-light ratio of a globular cluster. It is less popular nowadays because the computation of W can be carried out numerically, and because strip brightnesses are difficult to measure accurately in the presence of background stars.

2-11. [3] We have derived relations between the potential and surface density of non-axisymmetric disks by solving Laplace's equation in oblate spheroidal coordinates [see §2.6.4(a)] and cylindrical coordinates (Problem 2-9). Derive a relation of this kind by solving Laplace's equation in spherical coordinates, and show that the result is identical with the formula Kalnajs derived using logarithmic spirals [eq. (2-189)]. [Hint: You may need associated Legendre functions $P_\lambda^m(x)$, where λ is a complex number. Equations (1C-18) and (1C-7) may also be helpful.]

2-12. [3] Show that the circular speed $v_c(R)$ in a thin axisymmetric disk of surface density $\Sigma(R)$ may be written in the form (Mestel 1963)

$$v_c^2(R) = \frac{GM(R)}{R} + 2G\sum_{k=1}^{\infty}\alpha_{2k}\left[(2k+1)R^{-(2k+1)}\int_0^R \Sigma(R')R'^{2k+1}dR' - 2kR^{2k}\int_R^\infty \Sigma(R')R'^{-2k}dR'\right], \tag{2P-15}$$

where

$$\alpha_k = \pi\left[\frac{(2k)!}{2^{2k}(k!)^2}\right]^2. \tag{2P-16}$$

[Hint: Start with equation (2-139) and expand $|\mathbf{x}-\mathbf{x}'|^{-1}$ in Legendre polynomials using equation (1C-23).]

2-13. [3] (Suggested by H. Dejonghe) Prove that the surface density $\Sigma(x,y)$ and potential $\Phi(x,y)$ in a disk occupying the $z=0$ plane are related by

$$\Sigma(x',y') = \frac{1}{4\pi^2 G}\iint \frac{dx\,dy}{|\mathbf{x}-\mathbf{x}'|}\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right). \tag{2P-17}$$

(Hint: You may wish to use the results of §5.3.1.)