AST 353 Astrophysics — Problem Set 4

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I. VISCOSITY IN A PROTOSTELLAR ACCRETION DISK

In this question we would like to show that the viscosity in the accretion disk is

$$\nu_{vis} = \lambda_{\rm mfp} c_s \simeq r v_r, \tag{1}$$

as we have skimmed over in class. The basic equation is the angular momentum equation

$$\tau = \frac{dL}{dt}. \tag{2}$$

We follow the setup in Lecture 17 – consider an annulus with radius r, width dr, and thickness H, which is subject to a net viscous force F_{vis} due to differential rotation. We additionally assume the disk is thin, i.e., $H \ll R$, where R is the radius of the disk. Below we will provide a solution based on some order-of-magnitude arguments used in class, and a more exact one.

I.1. Quick Solution

In class we express F_{vis} using proportionality arguments,

$$\begin{aligned} F_{vis} &\simeq \nu_{vis} \rho \frac{v_{rot}}{d} \Delta A \\ &\simeq \nu_{vis} \frac{\Sigma}{H} \frac{v_{rot}}{r} 2\pi r H \\ &= 2\pi \nu_{vis} \Sigma v_{rot}, \end{aligned}$$

and the angular momentum of the annulus is

$$\begin{aligned} L &\simeq \Delta m \cdot v_{rot} r \\ &\simeq 2\pi r dr \Sigma \cdot v_{rot} r, \end{aligned} \tag{3}$$

where Σ is the surface density of the annulus. Assuming the rotational velocity v_{rot} does not vary significantly in time, we can write

$$\frac{dL}{dt} \simeq 2\pi r dr \Sigma \cdot v_{rot} v_r.$$

Substituting into Equation (2), we have

$$\begin{aligned} \tau = r F_{vis} &= \frac{dL}{dt} \\ 2\pi r \nu_{vis} \Sigma v_{rot} &= 2\pi r dr \Sigma \cdot v_{rot} v_r \\ \nu_{vis} &= dr v_r \simeq r v_r, \end{aligned} \tag{4}$$

with the last equality follows from the approximation $dr \simeq r$.

I.2. More Exact Solution

If you don't mind doing the more tedious algebra, we can approach the question more exactly. The annulus setup is the same. Recall that we assume Keplerian rotation for the disk, i.e., the central protostar's mass is much higher than the mass of the gas in the disk. For Keplerian disk we have

$$\Omega(r) = \sqrt{\frac{GM}{r^3}}, \tag{5}$$

and

$$v_{rot}(r) = r\Omega(r) = \sqrt{\frac{GM}{r}}. \tag{6}$$

The disk setup is most conveniently represented in the cylindrical coordinate system. In the cylindrical coordinate the equation of mass conservation reads

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho v_r\right) = 0.$$

Integrating over the z direction,

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}\left(r\Sigma v_r\right) = 0, \tag{7}$$

where both Σ and v_r are in general functions of both r and t. Similarly, the z-integrated equation of angular momentum conservation reads

$$r\frac{\partial}{\partial t}\left(r^2\Omega\Sigma\right) + \frac{\partial}{\partial r}\left(r^3\Omega\Sigma v_r\right) = \frac{1}{2\pi}\cdot\frac{\partial \tau(r)}{\partial r}, \tag{8}$$

where the term on the right hand side represents the net torque on the annulus, and $\tau(r)$ is given by

$$\tau(r) = 2\pi r \nu_{vis}\Sigma r^2 \cdot \frac{d\Omega}{dr}. \tag{9}$$

Combining Equation (7) and (8), and substituting Ω from Equation (5) and τ from Equation (9),

$$v_r(r) = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\nu_{vis}\Sigma\sqrt{r}\right). \tag{10}$$

Up to this point the equations are exact, so Σ from the last equation is still a function in both r and t. Now we approximate the partial derivative to arrive at the final answer (recall that with sloppy calculus we can write $\partial f/\partial x \sim f/x$)

$$|v_r(r)| \simeq \frac{3}{\Sigma\sqrt{r}}\cdot\frac{\nu_{vis}\Sigma\sqrt{r}}{r} \simeq \frac{\nu_{vis}}{r}.$$

II. FORMATION OF THE FIRST STARS

The key difference is the chemical composition of the gas from which the first-generation (Population III) stars formed – the primordial gas clouds were metal-free, containing no elements heavier than helium. The lack of metals as efficient coolants kept the primordial gas temperature at a much higher temperature of $\sim 200\,\mathrm{K}$, in contrast to $\sim 10\,\mathrm{K}$ in present-day star-forming clouds. To first order, the mass of a star scales as the Jeans mass of its parent gas cloud $M_J \propto T^{3/2}$. A higher temperature therefore points to a much more massive stellar population.