AST 353 Astrophysics — Problem Set 2

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I. SPHERICAL COLLAPSE

This question looks tough at first sight. If we disect it and relate the pieces to the concepts we already acquired in class, it could really be solve easily.

Let's start by restating the problem. At an initial redshift of $z_i = 1,000$, a spherical region is 1% denser ($\delta_i = 10^{-2}$) than the background universe. We assume the background universe is the one in PS1, part 1: flat ($k = 0$), containing both matter and dark energy, with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. An overdensity of 1% is huge, so we expect the spherical region to stop expanding, detach from the expanding background universe, and collapse on itself. Our task is to figure out when the collapse will happen, in terms of redshift z_{vir}.

We can think of the overdense region as a closed 'sub-universe' embedded in the flat expanding universe. The current problem stages in the matter-dominated era ($1 \lesssim z \lesssim 1000$), therefore we can use the approximate solution from PS1, part 1 (c)

$$t \simeq \frac{2(1+z)^{-3/2}}{3H_0\sqrt{\Omega_m}} = \frac{2a^{3/2}}{3H_0\sqrt{\Omega_m}} \tag{1}$$

to compute z_{vir} once we have the time of collapse t_{vir}. It is worth-clarifying that we should keep the Ω_m term in the expression. Using $\Omega_m = 1$ means an Einstein-de Sitter universe, which is different from the "$\Omega_m = 0.3; \Omega_\Lambda = 0.7$" universe in PS1 with the dark energy contribution dropped at early times.

Using the definition of overdensity we can further write down the initial density of the spherical region ρ_i,

$$\begin{aligned} \rho_i &= (1+\delta_i)\bar{\rho}(z = 1000) & (2)\\ &\simeq (1+\delta_i)\bar{\rho}_{m,0}(1+z_i)^3 & (3)\\ &\simeq 2.8 \times 10^{-21}\,\mathrm{g\,cm^{-3}} \end{aligned}$$

Also, in part 2 (c) of PS1, we solved for the time of Big Crunch t_c of a closed universe,

$$t_c = \sqrt{\frac{3\pi}{8G\rho_0}}, \tag{4}$$

where ρ_0 is the 'present-day' density of the closed universe. Our task remaining is to figure out what ρ_0 should be for the 'sub-universe' representing the overdense region.

In the following I will show you three different ways of solving it – from a very rough estimate, to the shortcut we discussed in class, to the exact analytical solution. Even if you solved it, it would be helpful to look at the other ways. It is always good to know multiple ways of solving the same problem.

I.1. Rough Estimate

We can directly write down a rough estimate if you recall that overdensity δ in an expanding flat universe grows with a power-law in time (Lecture 8),

$$\delta(t) = At^{2/3},$$

combining with Equation (1) we have

$$\delta \propto a$$
$$\Rightarrow a = a_i \frac{\delta}{\delta_i} \tag{5}$$

When the overdense region collapse $\delta_{vir} \simeq 2$, so

$$z_{vir} \sim a_{vir}^{-1} \sim a_i^{-1} \frac{\delta_i}{\delta_{vir}} \sim z_i \frac{\delta_i}{\delta_{vir}} \sim 1000 \cdot \frac{10^{-2}}{2} \sim 5.$$

Here we simply extrapolate from the linear growth regime. It may not be good enough, but it is not *too* far.

I.2. Approximate Solution

In Lecture 8 we found that ρ_0 should correspond to ρ_{ta}, the density at 'turn-around'. It was further approximated using mass conservation and Equation (5) as

$$\rho_{ta} \simeq \rho_i \delta_i^3 \tag{6}$$
$$\simeq 2.8 \times 10^{-27} \mathrm{g\,cm^{-3}}.$$

Equation (4) gives

$$t_{vir} \simeq 7.9 \times 10^{16}\,\mathrm{s} \simeq 2.5\,\mathrm{Gyr}.$$

Finally Equation (1) gives

$$\boxed{z_{vir} \simeq 2.6.}$$

I.3. Exact Solution

At this point we are not too far from the exact solution. Recall for the flat background universe

$$H^2(z) = \left(\frac{\dot{a}}{a}\right)^2$$
$$= \frac{8\pi G}{3}\bar{\rho}$$
$$H^2 \simeq \frac{8\pi G}{3}\bar{\rho}_m \quad \text{(ignoring dark energy).} \tag{7}$$

At initial time,

$$H_i^2 \simeq \frac{8\pi G}{3}\bar{\rho}_{m,i}, \tag{8}$$

where $\bar{\rho}_{m,i}$ is the initial matter density of the background universe.

We can then express the term GM at the initial time as

$$GM = G\frac{4\pi}{3}a_i^3\rho_i$$
$$\simeq G\frac{4\pi}{3}a_i^3(1+\delta_i)\bar{\rho}_{m,i}$$
$$\simeq \frac{1}{2}H_i^2 a_i^3(1+\delta_i). \tag{9}$$

By energy conservation, the initial total specific energy must be equal to that at turnaround,

$$\begin{aligned}
\frac{1}{2}v_i^2 - \frac{GM}{a_i} &= -\frac{GM}{a_{ta}} \\
\frac{1}{2}(H_i a_i)^2 - \frac{1}{2}H_i^2 a_i^2(1+\delta_i) &= -\frac{1}{2a_{ta}}H_i^2 a_i^3(1+\delta_i) \\
1-(1+\delta_i) &= -\frac{a_i}{a_{ta}}(1+\delta_i) \\
\Rightarrow \frac{a_i}{a_{ta}} &= \frac{\delta_i}{1+\delta_i}
\end{aligned} \tag{10}$$

Again using mass conservation,

$$\begin{aligned}
\rho_{ta} &= \left(\frac{a_i}{a_{ta}}\right)^3 \rho_i \\
&= \left(\frac{\delta_i}{1+\delta_i}\right)^3 \rho_i \\
&\simeq 2.7 \times 10^{-27} \mathrm{g\,cm^{-3}} \\
\Rightarrow t_{vir} &\simeq 8.1 \times 10^{16}\ \mathrm{s} \simeq 2.6\,\mathrm{Gyr}
\end{aligned}$$

The final answer is almost exactly equal to the above approximation,

$$\boxed{z_{vir} \simeq 2.6.}$$

II. WIMPS

Consider the dark matter halo of our Milky Way, it has a total mass of $10^{12}\,M_\odot$ and radius $R = 100\,\mathrm{kpc}$. Our Galaxy has formed at the center of this halo, we can safely assume that it has virialized.

At Virial equilibrium, we can assume all the WIMPs are moving more or less with an average velocity v_{vir}, and we have

$$2E_{\mathrm{kin}} = E_{\mathrm{pot}} \tag{11}$$

$$2 \cdot \frac{1}{2}Mv_{\mathrm{vir}}^2 = \frac{GM^2}{R} \tag{12}$$

$$v_{\mathrm{vir}} = \sqrt{\frac{GM}{R}} \tag{13}$$

Note that we are expressing the **total** kinetic energy and the **total** gravitational potential energy of the halo, not just for one WIMP. Substituting the values for M and R gives

$$\boxed{v_{\mathrm{vir}} \simeq 210\,\mathrm{km\,s^{-1}}}. \tag{14}$$