AST 353 Astrophysics — Problem Set 1

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I. COSMIC EXPANSION HISTORY

(a) Show that the solution to the Friedmann equation can be written as:

$$t(z) = \frac{1}{H_0} \int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}}\,, \tag{1}$$

where $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 70$ km s^{-1} Mpc^{-1}, and $t(z)$ is the age of the universe.

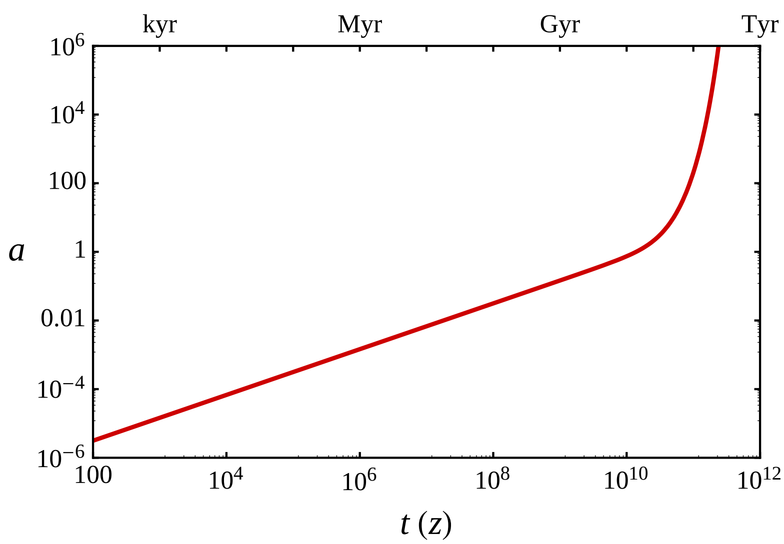
In class, we derived the Friedmann equation using Newtonian cosmology.

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_m a^{-3} + \Omega_\Lambda}\,.$$

We can substitute $a = (1+z')^{-1}$ and $\dot{a}dt = da = \frac{da}{dz'}dz' = -(1+z')^{-2}dz'$ to arrive at

$$\begin{aligned} t(z) &= \int_0^{t(z)} dt \\ &= \int_0^{a(z)} \frac{da}{a} \frac{1}{H_0\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} \\ &= \int_\infty^z \frac{-(1+z')^{-2}dz'}{(1+z')^{-1}} \frac{1}{H_0\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}} \\ &= \frac{1}{H_0} \int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}}\,. \end{aligned}$$

(b) Now, evaluate Eq. 1 numerically, and plot the result with $a = (1+z)^{-1}$ on the y-axis and time t on the x-axis. Use a log-log scaling and choose the time units appropriately.



(c) For the matter dominated era ($1 \lesssim z \lesssim 1000$) Eq. 1 can be solved analytically. Derive an approximate solution, $t(z) \approx \ldots$ for this situation.

To proceed we evaluate the integral dropping the Ω_Λ term:

$$\begin{aligned} t(z) &= \frac{1}{H_0}\int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m(1+z')^3+\Omega_\Lambda}} \\ &\approx \frac{1}{H_0}\int_z^\infty \frac{dz'}{\sqrt{\Omega_m}(1+z')^{5/2}} \\ &\approx \frac{1}{H_0\sqrt{\Omega_m}}\left[\frac{-2}{3}(1+z')^{-3/2}\right]_z^\infty \\ \Rightarrow &\quad \boxed{t(z) \approx \frac{2(1+z)^{-3/2}}{3H_0\sqrt{\Omega_m}}\,.} \end{aligned}$$

II. CLOSED UNIVERSE

Let us consider the closed case ($k = +1$), where the universe starts with a Big Bang, reaches a maximum expansion, turns around, and eventually ends in a Big Crunch.

For the closed model, it is convenient to write the Friedmann equation as follows:

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\left(a^{-1} - 1\right)\,, \tag{2}$$

where ρ_0 is the present-day mass density (in matter; we here assume there is no dark energy).

(a) Parametric Expressions

The Friedmann equation can be solved with the following parametric expressions:

$$a = \sin^2\alpha \qquad \text{and} \qquad t = A(\alpha - \sin\alpha\cos\alpha)\,,$$

where α is a "development angle", such that $\alpha = 0$ corresponds to the Big Bang, $\alpha = \frac{\pi}{2}$ corresponds to the point of maximum expansion ("turn-around"), and $\alpha = \pi$ to the Big Crunch.

To find the constant A we first find $\dot{a}$ from the parametric expressions. Consider that

$$\frac{da}{d\alpha} = \frac{d}{d\alpha}\left(\sin^2\alpha\right) = 2\sin\alpha\cos\alpha$$

and

$$\frac{dt}{d\alpha} = A\frac{d}{d\alpha}(\alpha - \sin\alpha\cos\alpha) = A\left(1 - \cos^2\alpha + \sin^2\alpha\right) = 2A\sin^2\alpha\,,$$

so the derivative becomes

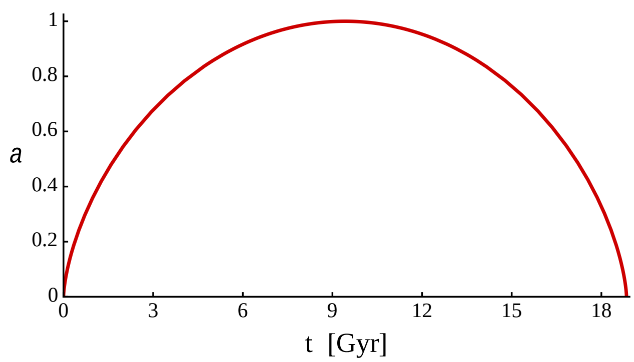
$$\dot{a} = \frac{da}{dt} = \frac{da/d\alpha}{dt/d\alpha} = \frac{2\sin\alpha\cos\alpha}{2A\sin^2\alpha} = \frac{1}{A\tan\alpha}\,.$$

Thus, according to Eq. 2 we have

$$
\begin{aligned}
0 &= \left(\frac{1}{A\tan\alpha}\right)^2 - \frac{8\pi G\rho_0}{3}\left(\frac{1}{\sin^2\alpha} - 1\right) \\
&= \frac{\cot^2\alpha}{A^2} - \frac{8\pi G\rho_0}{3}\left(\csc^2\alpha - 1\right) \\
&= \cot^2\alpha\left(\frac{1}{A^2} - \frac{8\pi G\rho_0}{3}\right) \\
\Rightarrow &\qquad \boxed{A = \sqrt{\frac{3}{8\pi G\rho_0}}\,.}
\end{aligned}
$$

(b) Plot

Plot this solution, showing the scale factor on the y-axis, and time on the x-axis. Assuming $\rho_0 = 5 \times 10^{-29}$ g cm^{-3}, show time in units of Gyr. (Recall that the scale factor is dimensionless.)



(c) Age of the Universe

What is the total duration of such a universe? I.e., what is the time elapsing between Big Bang and Big Crunch (in units of Gyr)?

We simply need to calculate the time when $\alpha = \pi$:

$$
t_{\text{Big Crunch}} = \sqrt{\frac{3}{8\pi G\rho_0}}(\alpha - \sin\alpha\cos\alpha)\Big|_{\alpha=\pi} = \sqrt{\frac{3\pi}{8G\rho_0}} = 18.84 \text{ Gyr}\,. \tag{3}
$$