AST353 (Spring 2016)

ASTROPHYSICS

Problem Set 1

Due in class: Thursday, February 4, 2016

(worth 10/100)

1. Cosmic Expansion History

In class, we derived the Friedmann equation. Assume a flat universe ($k = 0$).

a. Show that the solution to the Friedmann equation can be written as:

$$t(z) = \frac{1}{H_0} \int_z^\infty \frac{dz}{(1+z)\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} \ , \tag{1}$$

where $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 70\,\mathrm{km\ s^{-1}\,Mpc^{-1}}$, and $t(z)$ is the age of the universe at redshift z.

b. Now, evaluate equ. (1) numerically, and plot the result with $a = 1/(1+z)$ on the y-axis, and time, t, on the x-axis. Use a log-log scaling, and choose the time units appropriately (please ask Benny or me for help if needed). You can use a program like MATHEMATICA or MAPLE to do the numerical integration and the plotting; or whatever program(s) you are already familiar with.

c. For the matter dominated era ($1,000 \gtrsim z \gtrsim 1$, when dark energy can still be neglected, such that $\Omega_\Lambda \simeq 0$), equ. (1) can be solved analytically. Derive an approximate solution, $t(z) \simeq ...$ for this situation.

2. Closed Universe

Now, let us consider the Friedmann equation for the closed case ($k = +1$), where the universe starts with a Big Bang, reaches a maximum expansion, turns around, and eventually ends in a Big Crunch.

For the closed model, it is convenient to write the Friedmann equation as follows:

$$\dot{a}^2 = \frac{8\pi G \rho_0}{3} \left(a^{-1} - 1 \right) \ ,$$

where ρ_0 is the present-day mass density (in matter; we here assume there is no dark energy).

a. Show that this equation can be solved with the following parametric expressions:

$$a = \sin^2 \alpha \ ,$$

and

$$t = A\,(\alpha - \sin\alpha \cos\alpha) \ \ ,$$

where α is a "development angle", such that $\alpha = 0$ corresponds to the Big Bang, $\alpha = \pi$ corresponds to the point of maximum expansion ("turn-around"), and $\alpha = 2\pi$ to the Big Crunch.

Find an expression for the constant A.

b. Plot this solution, showing the scale factor on the y-axis, and time on the x-axis. Assuming $\rho_0 = 5 \times 10^{-29}\,\mathrm{g\,cm^{-3}}$, show time in units of Gyr. (Recall that the scale factor is dimensionless.)

c. What is the total duration of such a universe? I.e., what is the time elapsing between Big Bang and Big Crunch (in units of Gyr)?