Last time we discussed an idealized situation of structure formation, namely, growth of structure in a static medium.

— The final equation that governs the growth is:

$$\ddot{\delta} = 4\pi G \bar{\rho}\, \delta$$

— Two key points are recapped here:

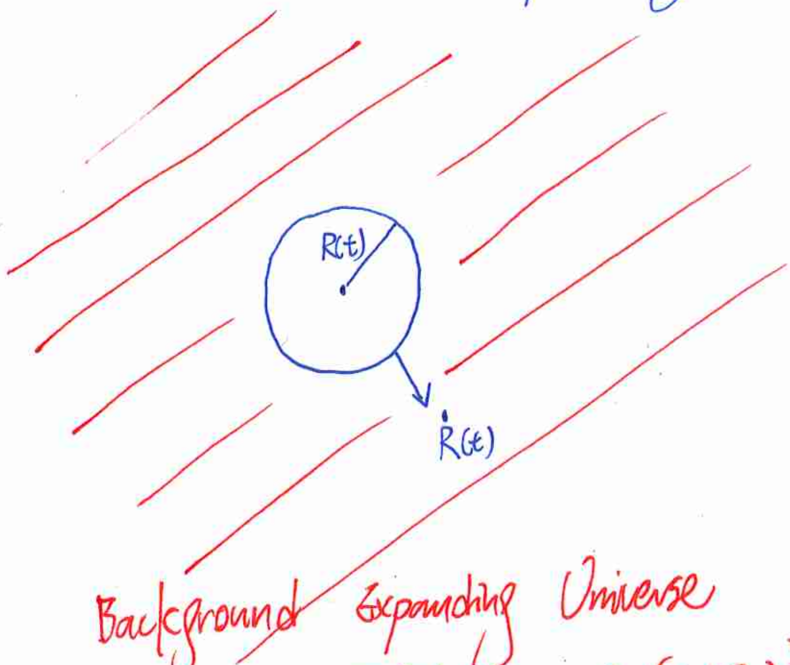
(1) the growth is exponential;

(2) the timescale of the growth is a VERY IMPORTANT timescale, the "free-fall" time t_{ff}.

$$t_{ff} \sim \frac{1}{\sqrt{G\bar{\rho}}}.$$

In this lecture, we continue the discussion of structure formation. This time we focus on a slightly more advanced system, where the structure forms from an expanding universe.

(ii) Growth in Expanding Universe



$$M = \frac{4}{3}\pi R^3 \rho = \text{constant}.$$

Background Expanding Universe

$$\bar{\rho}(t) = \rho_0 (1+z)^3 = \rho_0 a^{-3}$$ [Assuming a matter-dominated universe].

Consider an overdense region with radius $R(t)$ and mass M, for which we can express the overdense density as

$$\rho(t) = (1 + \delta(t))\,\bar{\rho}(t).$$

Please be reminded that no only $R(t)$ of the region is a fuction of time, but also the background density $\bar{\rho}(t)$.

Similar to what we did before, we write down the e.o.m. by considering a thin shell of mass ΔM:

$$\cancel{\Delta M} \cdot \ddot{R} = -\frac{GM}{R^2} \cdot \cancel{\Delta M}$$ [Just ma=F]

$$\Rightarrow \boxed{\ddot{R} = -\frac{4\pi G}{3} R\rho = -\frac{4\pi G}{3} R\bar{\rho} - \frac{4\pi G}{3} R\,\delta\bar{\rho}} \quad (1)$$

Recall for the background universe:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}}$$

Then consider conservation of mass,

$$M = \frac{4\pi}{3}\bar{\rho}(t)\cdot(1+\delta(t))\cdot R^3(t) = \text{constant}.$$

$$= \frac{4\pi}{3}\rho_0 a^{-3}(1+\delta)\cdot R^3 = \text{const}.$$

$$R(t) = \frac{3M}{4\pi\rho_0}\cdot(1+\delta)^{-\frac{1}{3}} = \tilde{k}\cdot a(1+\delta)^{-\frac{1}{3}},$$

where $\tilde{k} = \frac{3M}{4\pi\rho_0}$ is just another constant.

In equation ① we need a $\ddot{R}$ term, so take 2 time derivatives $\left(\frac{d^2}{dt^2}\right)$ of the above:

$$\ddot{R} = \tilde{k}\cdot a(1+\delta)^{-\frac{1}{3}}\left[\frac{\ddot{a}}{a} - \frac{2}{3}\cdot\frac{\dot{a}}{a}(1+\delta)^{-1}\dot{\delta} + \frac{4}{9}(1+\delta)^{-2}\delta^{-2} - \frac{1}{3}(1+\delta)^{-1}\ddot{\delta}\right].$$

Neglect 2nd order terms:

$$\boxed{\frac{\ddot{R}}{R} \cong \frac{\ddot{a}}{a} - \frac{2}{3}\cdot\frac{\dot{a}}{a}\dot{\delta} - \frac{1}{3}\ddot{\delta}} \qquad ②$$

→ Now combine ① and ② 02-16-2016.

$$\cancel{\frac{\ddot{a}}{a}} - \frac{2}{3}\cdot\frac{\dot{a}}{a}\cdot\dot{\delta} - \frac{1}{3}\ddot{\delta} = \cancel{-\frac{4\pi G}{3}\bar{\rho}} - \frac{4\pi G}{3}\delta\bar{\rho}$$

Terms cancel : e.o.m. of the background universe.

$$\Rightarrow \quad \ddot{\delta} + 2\cdot\frac{\dot{a}}{a}\cdot\dot{\delta} = 4\pi G\bar{\rho}\,\delta$$

Recall $H(t) = \dfrac{\dot{a}(t)}{a(t)}$,

$$\boxed{\ddot{\delta} + 2H\cdot\dot{\delta} = 4\pi G\bar{\rho}\,\delta}$$

Compare with the δ-equation from the static medium case, we have an extra $2H\dot{\delta}$ term now.
The term $\dot{\delta}$ is the "velocity" of the growth of the overdensity, so this extra term acts like a friction term to slow down the growth — "Hubble friction".

→ in general, solve numerically!

→ consider a simplified case of background universe, Einstein - de Sitter [EdS].

Recall from PS #1, 02-16-2016

$$H^2(t) = \left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\Omega_m (1+z)^3 + \Omega_\Lambda\right]$$

EdS universe: $\Omega_m = 1.0$; $\Omega_\Lambda = 0.0$.

$$H = \frac{\dot a}{a} = \frac{1}{a}\cdot\frac{da}{dt} = H_0 \cdot a^{-3/2}$$

It can be solved analytically by separation of variables.

$$a(t) = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3} \propto t^{2/3}$$

Note:

$$H = \frac{\dot a}{a} = \frac{2}{3t}$$

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\bar\rho$$

$$\Rightarrow \quad 4\pi G\bar\rho = \frac{3H^2}{2} = \frac{2}{3t^2}$$

Plug it in the δ-equation:

$$\boxed{\ddot\delta + \frac{4}{3t}\cdot\dot\delta = \frac{2}{3t^2}\delta} \quad [\text{EdS}]$$

It's an ODE for $\delta(t)$ with non-constant coefficients.

Try power-law ansatz.

$$\delta = A t^x$$

$$\Rightarrow \quad x(x-1)\cdot t^{x-2} + \frac{4}{3} x\, t^{x-2} = \frac{2}{3} t^{x-2}$$

$$\Rightarrow \quad x^2 + \frac{1}{3}x - \frac{2}{3} = 0$$

$$\Rightarrow \quad x = -\frac{1}{6} \pm \frac{5}{6} = \begin{cases} 2/3 \\ -1. \end{cases}$$

$$\therefore \quad \delta(t) = A t^{2/3} + \cancel{B t^{-1}}$$

again, ignore decaying mode.

$$\delta(t) = A t^{2/3} \propto t^{2/3} \propto a.$$

$z_i = 1000$

R_i $\qquad \delta_i = 0.01 = 10^{-2}$

$$M = \frac{4\pi}{3} \cdot R_i^3 \cdot \rho_i$$

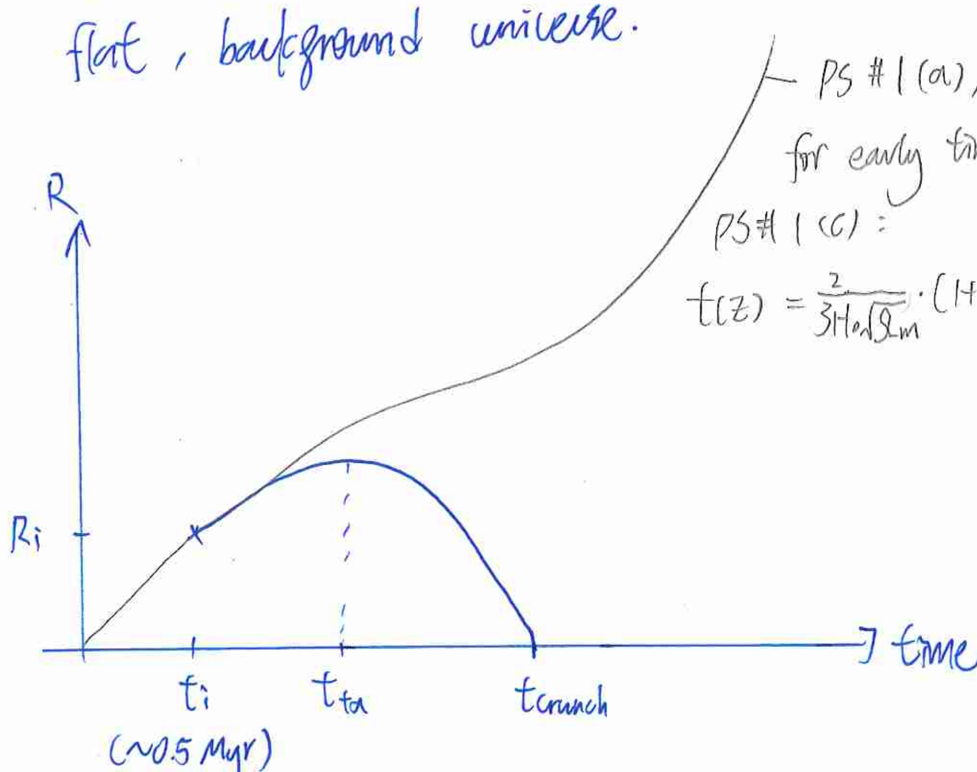
$$\rho_i = (1+\delta_i)\bar{\rho}(z=1000) \quad \text{[definition].}$$

$$= (1.01) \cdot \rho_{m,0} (1+z_i)^3$$

$$\simeq 2.8 \times 10^{-21}\ g\,cm^{-3}.$$

The dense region itself behaves as a closed "sub-universe" inside an expanding flat, background universe.

Schematically:



PS #1 (a),
for early time,
PS #1 (c):

$$t(z) = \frac{2}{3H_0\sqrt{\Omega_m}} \cdot (1+z)^{-3/2}$$

Recall in PS #1, question 2, 02-16-2016.

the Big-Crunch equation:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_0 (a^{-1} - 1)$$

$$t_{crunch} = \pi A = \sqrt{\frac{3\pi}{8\pi G\rho_0}}.$$

Big Q: How do we fix ρ_0 for our overdensity "sub-universe"?

A: Return to first principles.

Friedmann equation: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$

To fix k, consider the moment of turnaround!

at ta: $0 = \dot{a}^2 = \frac{8\pi G}{3}\rho_{ta} \cdot a_{ta}^2 - k$

Mass conservation: $\rho a^3 = \rho_{ta} \cdot a_{ta}^3$

Plug in FE: $\dot{a}^2 = \frac{8\pi G}{3}\rho_{ta} \cdot a_{ta}^3 \left(\frac{1}{a} - \frac{1}{a_{ta}}\right)$

Set $a_{ta} = 1$,

$$\dot{a} = \frac{8\pi G}{3}\rho_{ta} (a^{-1} - 1)$$

☆ Now use ρ_{ta} as ρ_0 in PS #1, Q2.

Final job: what is ρ_{ta}? 02-16-2016

$$\rho_{ta} = \rho_{ta} \cdot a_{ta}^3 = \rho_i \cdot a_i^3$$

$$\delta \propto a \Rightarrow \delta = \delta_i \cdot \frac{a}{a_i}$$

$$\Rightarrow 1 \cong \delta_{ta} = \delta_i \cdot \frac{a_{ta}}{a_i} = \frac{\delta_i}{a_i}$$

$$\Rightarrow a_i \sim \delta_i$$

$\therefore \rho_{ta} \simeq \rho_i \cdot \delta_i^3$, while you know both ρ_i and δ_i.