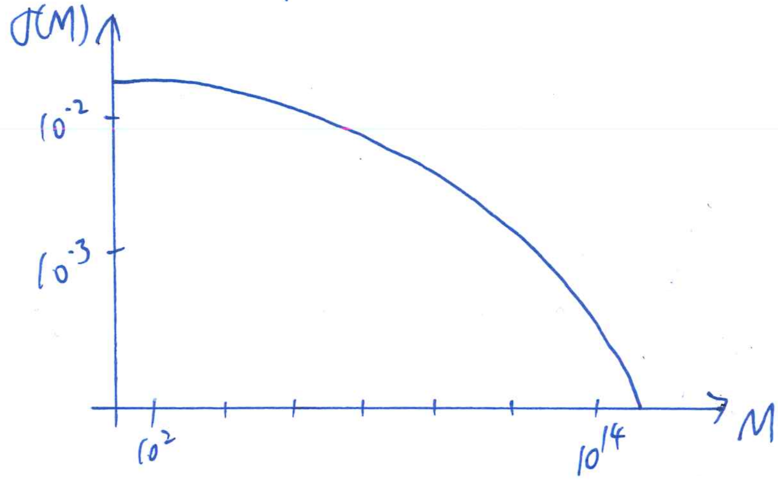
Last time we showed the fluctuation at different mass scales, which is reproduced here:



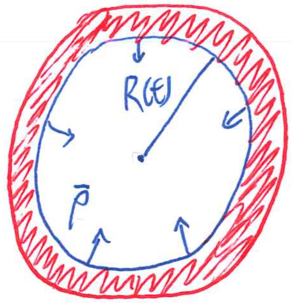
Here we re-iterate that an important message from this plot is the "bottom-up" structure formation scheme.

In this lecture, we will focus on the second question we posted in Lecture 5:

② How do the perturbations grow?
[a.k.a. "gravitational instability"].

(i) Idealized case: growth in a static medium.
[an actual example of this case can be found in star formation in the MW.]

Consider a spherical object with a fixed mass M, initially at rest, as time goes by its radius may change, 02-11-2016



ΔM : mass of an infinitesimal shell of mass.

$$M = \frac{4}{3}\pi R^3 \cdot \bar{\rho}$$

$\bar{\rho}$: density of the object.
(this is not the background density of the object. we are only considering one object, which has density of $\bar{\rho}$.)

Write down the e.o.m. of the shell,

$$\Delta M \cdot \ddot{R} = -\frac{GM \cdot \Delta M}{R^2}$$

$$\Rightarrow \quad \ddot{R} = -\frac{GM}{R^2}$$

$$\Rightarrow \quad \ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi}{3} \cdot G \cdot \bar{\rho} \cdot R$$

Now, perform a thought experiment (Einstein called it "Gedanken experiment").

Imagine we have a force F which balances the gravity, the object will be stable and be like that forever.

In astrophysics, this force may be pressure, magnetic field, etc.

Then we have, 02-11-2016

$$\ddot{R} = -\frac{4\pi G}{3} \cdot \bar{\rho} \cdot R + \frac{F}{\Delta M} = 0$$

Now, what if we perturb the density in the spherical object?

$$\text{new } \bar{\rho} = \bar{\rho}(1+\delta) = \bar{\rho} + \bar{\rho}\delta$$

The new e.o.m. for the perturbed object becomes

$$\ddot{R} = -\frac{4\pi G}{3} \underbrace{\bar{\rho}(1+\delta)}\cdot R + \underbrace{\frac{F}{\Delta M}}$$

gravitational force increases a little because of the perturbation. ($1 \gg \delta > 0$)

Same balancing force, the perturbation is small.

$$\Rightarrow \quad \ddot{R} = -\frac{4\pi G}{3} \cdot \bar{\rho}\,\delta R \quad \text{———} \quad (1)$$

Note: we have one equation and two unknowns: R and δ.

→ we need a second equation.

→ Conservation of mass 02-11-2016

(we don't lose mass as the object collapses).

$$M = \frac{4\pi}{3} \cdot R^3 \bar{\rho} (1+\delta) = \text{constant}.$$

Take $\frac{d}{dt}$,

$$0 = \dot{M} = \frac{4\pi}{3}\bar{\rho} [3R^2 \dot{R} (1+\delta) + R^3 \dot{\delta}]$$

$$\Rightarrow \quad 0 = 3R^2 \cdot \dot{R} (1+\delta) + R^3 \dot{\delta}$$

$$\Rightarrow \quad 0 = 3\dot{R} (1+\delta) + R\dot{\delta} \qquad (\text{cancelling } R^2)$$

Take another $\frac{d}{dt}$,

$$0 = 3\ddot{R} (1+\delta) + 4\dot{R}\dot{\delta} + R\ddot{\delta}$$

use (i) $\delta \ll 1$, drop δ in the first term.

(ii) $\dot{R} \approx 0$ initially, drop $\dot{R}$ in the second term.

$$\Rightarrow \quad 0 = 3\ddot{R} + R\ddot{\delta} \quad \text{———} \quad (2)$$

Combine (1) & (2):

$$\ddot{\delta} = 4\pi G \bar{\rho} \delta \qquad \boxed{\text{Jeans}}$$

[Now one equation one unknown.]

The equation can be solved using exponential functions: 02-11-2016

$$\delta = A e^{\lambda t} + \cancel{B e^{-\lambda t}} \quad \text{(general solution)}$$

$$\Rightarrow \quad \ddot{\delta} = \lambda^2 \delta \, , \quad \text{where} \quad \lambda^2 = 4\pi G \bar{\rho}$$

In face of structure formation (δ should be increasing), we drop the negative exponential (decaying) term in the general solution.

$$\delta = A e^{\lambda t} = \delta_i \, e^{\lambda t}$$

$\Rightarrow$ exponential growth!

Final question of today:

How quick is the instability?

$$\delta_i (t_e) = \delta_i \cdot e = \delta_i \, e^{\lambda t_e}$$

t_e: e-folding time

$$1 = \lambda t_e$$

$$\Rightarrow \quad t_e = \frac{1}{\lambda} = \sqrt{\frac{1}{4\pi G \bar{\rho}}} = t_{ff} \quad \text{"free-fall" time.}$$

This is a VERY IMPORTANT timescale in astrophysics!!!