In this lecture we pick up from last time 02-23-2016.
to continue the derivation of ρ_{vir}, the density of the dark matter halo at virialization.

Recall that we are trying to gain so insight by considering the Einstein-de Sitter model. We have:

$$a \propto t^{2/3}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t}$$

$$\Rightarrow \quad \bar{\rho}(t) = \frac{1}{6\pi G t^2}$$

We wish to know what the density of the halo is, so we start by writing

$$\rho_{halo}(t) = \rho(t) = M \Big/ \tfrac{4}{3}\pi R_{halo}^3$$

$$= M \Big/ \tfrac{4}{3}\pi R^3.$$

Similar to the problem sets we did, we will borrow the Big-Crunch solution:

$$a = \sin^2\alpha$$

$$t = A(\alpha - \cos\alpha \sin\alpha), \quad A = \sqrt{\frac{3}{8\pi\rho_0 G}}$$

However, the Big-Crunch model describes the whole universe. We have to re-dimensionalize the scale factor. 02-23-2016

we know $R \propto a$

or $R = R_{ta} \cdot a = R_{ta} \cdot \sin^2\alpha$.

(R_{ta}: radius at turn-around, which has dimension of length, eg., cm, pc, ...)

Q: How is R_{ta} connected to R_{vir} ?

A: $\longrightarrow$ use energy conservation.

$$E_{tot} = E_{pot} + E_{kin} = \text{constant}.$$

At turnaround: $E_{tot} = E_{pot}$ $\quad$ ($E_{kin} = 0$)

$$= -\frac{GM^2}{R_{ta}}$$

At Virialization: $E_{tot} = \frac{1}{2} E_{pot}$ $\quad$ (Virial theorem).

$$= -\frac{GM^2}{2R_{vir}}$$

$$\Rightarrow \quad R_{vir} = \frac{1}{2} \cdot R_{ta}.$$

Then $\rho(t) = \frac{M}{\frac{4}{3}\pi R^3} = \frac{M}{\frac{4}{3}\pi R_{ta}^3} \cdot (\sin\alpha)^{-6}$ 02-23-2016

$= \rho_{ta} \cdot (\sin\alpha)^{-6}$. for $0 \leq \alpha \leq \pi$.

Background universe:

$$\bar{\rho} = \frac{1}{6\pi G t^2} = \frac{1}{6\pi G A^2} \cdot (\alpha - \cos\alpha \cdot \sin\alpha)^{-2}$$

$$= \frac{4}{9} \cdot \rho_{ta} (\alpha - \cos\alpha \sin\alpha)^{-2}$$

$$\Rightarrow \frac{\rho(t)}{\bar{\rho}(t)} = \frac{9}{4} \cdot \frac{(\alpha - \cos\alpha \sin\alpha)^2}{(\sin\alpha)^6}$$

Evaluate at turnaround $\alpha = \frac{\pi}{2}$.

$$\frac{\rho(t=t_{ta})}{\bar{\rho}(t=t_{ta})} = \frac{9}{4} \cdot \frac{\pi^2}{4} = \frac{9\pi^2}{16}$$

$$\Rightarrow \rho_{ta} = \rho(t=t_{ta})$$

$$= \frac{9\pi^2}{16} \cdot \bar{\rho}(t=t_{ta})$$

$$= 5.6 \cdot \bar{\rho}(t=t_{ta})$$

→ Final step:

$$\rho_{vir} = M / \tfrac{4}{3}\pi R_{vir}^3 = 8 \rho_{ta}$$

But, background has expanded since turnaround.

$$t_{vir} = 2 t_{ta}$$

Remember $\quad \bar{\rho}(t) \propto \frac{1}{t^2}$

$$\bar{\rho}(t = t_{vir}) = \tfrac{1}{4} \bar{\rho}(t = t_{ta})$$

$$\Rightarrow \quad (\rho / \bar{\rho})_{vir} = 32 \left(\frac{\rho}{\bar{\rho}}\right)_{ta}$$

$$= 18\pi^2$$

$$\simeq 178 \simeq 200$$

$$\rho_{vir} = 200 \bar{\rho}$$