AST353 Group Project 2 — Review Session. 04-18-2016.

Hints: Paper by Peters (1964)

Physical Review, vol. 136, page 1224.

In short, using the weak-field approximation for general relativity and after pages of derivation, we have the following equations that govern the time evolution of semi-major axis 'a' and the eccentricity 'e' for the orbit of two point masses emitting gravitational wave:

$$\left\{\begin{aligned} \frac{da}{dt} &= -\frac{64}{5}\cdot\frac{G^3 M_1 M_2 (M_1+M_2)}{c^5 a^3 (1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \quad &\text{—— (1)} \\ \frac{de}{dt} &= -\frac{304}{15}\cdot\frac{G^3 M_1 M_2 (M_1+M_2)}{c^5 a^4 (1-e^2)^{5/2}}\cdot e\cdot\left(1+\frac{121}{304}e^2\right) \quad &\text{—— (2)} \end{aligned}\right.$$

$$\frac{(1)}{(2)}: \quad \frac{da}{de} = \frac{12}{19}\cdot\frac{a}{e}\cdot\frac{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{(1-e^2)\left(1+\frac{121}{304}e^2\right)} \quad \text{—— (3)}$$

③ can be integrated to find $a(e)$, 04-18-2016

$$a(e) = \frac{c_0 e^{12/19}}{(1-e^2)} \left(1 + \frac{121}{304} e^2\right)^{870/2299},$$

where $c_0 = \frac{a_0 (1-e_0^2)}{e_0^{12/19}} \cdot \left(1 + \frac{121}{304} e_0^2\right)^{-870/2299}$ is a constant fixed by the initial a_0 and e_0.

By setting $\beta = \frac{64}{5} \cdot \frac{G^3 M_1 M_2 (M_1 + M_2)}{c^5}$, we can rewrite ②,

$$\frac{de}{dt} = -\frac{19}{12} \cdot \frac{\beta}{c_0^4} \cdot \frac{e^{-29/19} (1-e^2)^{3/2}}{\left[1 + \frac{121}{304} \cdot e^2\right]^{1181/2299}}. \quad \text{—— ④}$$

Since $e \to 0$ as $a \to 0$, we can find the inspiral time t_{insp} by integrating ④

$$t_{insp} = \frac{12}{19} \cdot \frac{c_0^4}{\beta} \cdot \int_0^{e_0} \frac{e^{29/19} \left(1 + \frac{121}{304} e^2\right)^{1181/2299}}{(1-e^2)^{3/2}} \, de,$$

which can be evaluated numerically.

Assuming that the black hole (BH) binary system was formed at redshift $z_0 = 20.0$; 04-18-2016

LIGO observation found that the merger occurred at $z_{merger} = 0.088 \sim 0.1$.

By picking a reasonable initial eccentricity e_0, we can solve for $\underline{a_0}$ numerically:

$$\underbrace{t(z_{merger})}_{\text{known}} - \underbrace{t(z_0)}_{\text{assumed}} = t_{insp}(\underbrace{a_0}_{\text{find.}}, \underbrace{e_0}_{\text{picked}})$$

Note: If we set $e_0 = 0$, we can write

$$t_{insp} = a_0^4 / 4\beta.$$

If we consider e_0 to be small,

$$t_{insp} \simeq \frac{12}{19} \cdot \frac{c_0^4}{\beta} \int_0^{e_0} e^{29/19} de = \frac{c_0^4}{4\beta} \cdot e_0^{48/19}$$

- Both can be solved without numerical integration.