Review session — PS #3., Question 1. 02-29-2016.

#1. Our aim here is to solve the equation for the overdensity evolution:

$$\ddot{\delta}(t) + 2H(t)\cdot\dot{\delta}(t) = 4\pi G\cdot\bar{\rho}(t)\cdot\delta(t) \quad (*)$$

This is a second-order ODE for $\delta(t)$ (the unknown). The other functions — $H(t)$ and $\bar{\rho}(t)$ — are known from PS#1.

We solved the above equation analytically for the simple case of an Einstein-de Sitter (E-dS) background universe (see Lecture 8 notes). The solution reads

$$\delta_{EdS}(t) = A\,t^{2/3}.$$

With the initial condition ($\delta_i = 10^{-4}$ at $z_i = 1000$), we can write

$$\delta_{EdS} = \delta_i\left(\frac{t}{t_i}\right)^{2/3},$$

where t_i is the time corresponds to z_i.

This function will be useful in part (c).

In order to solve (*) numerically, we need to provide the numerical solver ("NDSolve" in Mathematica). 02-29-2016

(i) The equation (*)

(ii) All necessary functions : $H(t)$ and $\bar{\rho}(t)$.

(iii) Two initial conditions ($\because$ 2nd order ODE)

(i) We already have (*).

(ii) From PS #1, we have

$$\begin{cases} H(z) = \dfrac{\dot{a}}{a} = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda} \\ \bar{\rho}(z) = 3H^2(z) / 8\pi G. \end{cases}$$

They are functions of z, not t. We can "transform" them into functions of t by constructing a function $z(t)$.

Then,
$$\begin{cases} H(t) = H(z(t)) \\ \bar{\rho}(t) = \bar{\rho}(z(t)) \end{cases}$$

Recall from PS #1, we numerically evaluated

$$t(z) = \frac{1}{H_0} \cdot \int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}.$$

We can numerically construct a "Look-up table" for z given a t value, which works exactly as $z(t)$. This can be done using the "Interpolation" method in Mathematica.

iii) For initial conditions,

$$① \quad \delta(t(z_i)) = \delta_i = 10^{-4}$$

$$② \quad \dot{\delta}(t(z_i)) = \dot{\delta}_{EdS}(t(z_i))$$

Schematically, the line of Mathematica code for solving the equation looks like:

actual ODE

$$\text{NDSolve}[\boxed{\ddot{\delta}(t) + 2H(t)\cdot\dot{\delta}(t) - 4\pi G\bar{\rho}(t)\,\delta(t) == 0},$$

initial conditions

$$\boxed{\delta(t(z_i)) == \delta_i, \quad \dot{\delta}(t(z_i)) == \dot{\delta}_{EdS}(t(z_i))},$$

$$\boxed{\delta}, \boxed{\{t, t_{min}, t_{max}\}}]$$

function to be solved

Range of independent variable concerned

★★ We use a double equal sign "==" to denote equations to be solved in numerical solvers of Mathematica.