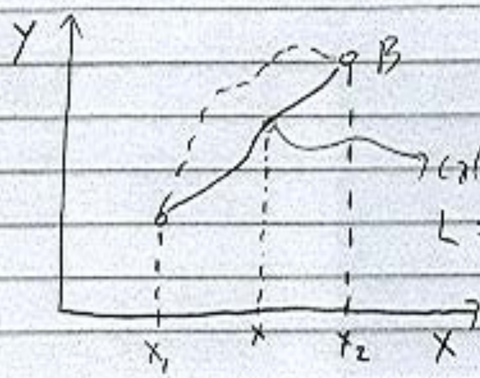
Calculus of Variations: Brief Introduction

(1) Basic Method:

- Consider paths in x-y plane from point A to B:



calculate value of some function $L = L(y, y')$ for each x along the path !

$$\left(y' \equiv \frac{dy}{dx} \right)$$

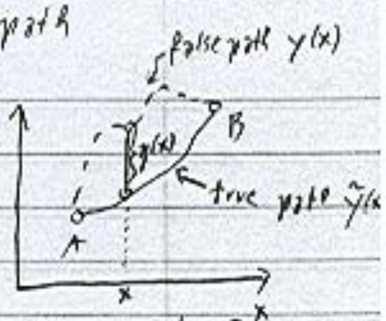
- Take integral along path

$$\boxed{I = \int_{x_1}^{x_2} L(y(x), y'(x))\, dx}$$

- For different paths $y(x)$, one gets different values of I

- Ask: For what path $\tilde{y}(x)$ is integral minimum (or maximum) → in general: an extremum ?

- Consider difference between (unknown) true path and a nearby path:



$$\boxed{y(x) = \tilde{y}(x) + \eta(x)}$$

$$\left[\Rightarrow y'(x) = \tilde{y}'(x) + \eta'(x) \quad , \text{ where } \eta'(x) = \frac{d\eta}{dx} \right]$$

→ Here $\eta(x)$ is an arbitrary function, which is very small compared to $\tilde{y}(x)$ $\left[|\eta(x)| \ll |\tilde{y}(x)| \right]$

- now evaluate the difference:

$$\Delta I = \int_{x_1}^{x_2} \left[L(y, y') - L(\tilde{y}, \tilde{y}') \right] dx$$

[Note: in the calculus of variations, ΔI is often denoted δI]

- Taylor-expand:

$$L(y, y') = L(\tilde{y}, \tilde{y}') + \frac{\partial L}{\partial y} \eta(x) + \frac{\partial L}{\partial y'} \eta'(x) + (\text{higher-order terms in } \eta)$$

- Insert:

$$\Delta I = \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial y} \eta(x) + \frac{\partial L}{\partial y'} \eta'(x) \right] dx$$

- Use integration by parts to rewrite $\int \frac{\partial L}{\partial y'} \eta'(x)\, dx$:

→ Since $\eta'(x) = \frac{d}{dx}\eta(x)$, one finds:

$$\int_{x_1}^{x_2} \frac{\partial L}{\partial y'} \eta'(x) = \cancel{\left[\frac{\partial L}{\partial y'}\eta(x)\right]_{x_1}^{x_2}}^{\,\to 0} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\, dx$$

- Now comes the crucial trick:

→ All paths (true or false) must go through points A and B

$$\Rightarrow \quad \eta(x_1) = \eta(x_2) = 0$$

⇒ The first term on RHS is ZERO!

- We can now write:

$$\Delta I = \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] \eta(x)\, dx$$

- To have an extremum, need: $\Delta I = 0$

(recall ordinary calculus!)

- But: Integral above is zero for arbitrary $\eta(x)$ ONLY IF

$$\boxed{\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}}$$

(This is famous <u>Euler-Lagrange equation</u>)

- One can make this more general:

  → assume that function L depends on more than one variable:

$$L = L(y_1, y_1', y_2, y_2', y_3, y_3', \ldots) = L(y_i, y_i')$$

  → One then has an equation for each pair (y_i, y_i'):

$$\boxed{\frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right) = \frac{\partial L}{\partial y_i}} \quad \text{for all } i$$

- Note: Be very careful when you take the partial derivatives, in terms of what to hold constant. E.g., when you take $\frac{\partial L}{\partial y_i}$, make sure to treat y_i' as a constant and vice versa !!

  Be also very careful when you apply the $\frac{d}{dx}$ derivative. Both y_i AND y_i' depend on x !

(2) An Example:

→ The most famous of them all:

THE BRACHISTOCHRONE[*] PROBLEM

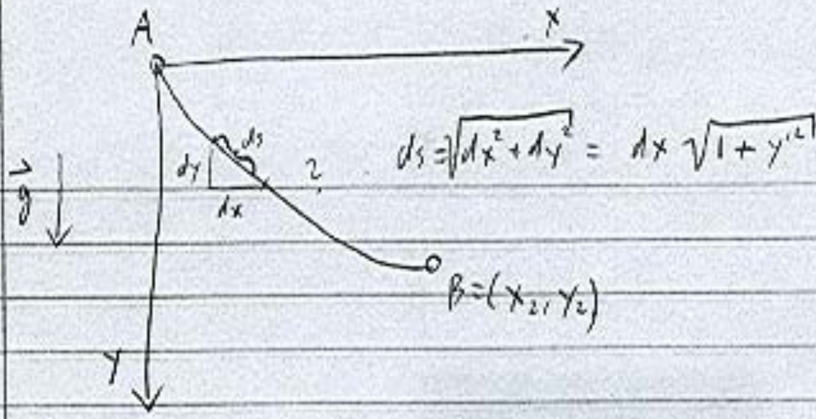
(* Greek for "shortest time")

— posed in 1696 by Johann Bernoulli:

> "Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument."

— Problem: Consider a particle with mass m, which starts from rest at point A and falls under the influence of Earth's gravity, assuming that the particle is restricted to fall along a frictionless curve $y(x)$ from the origin at $A = (0,0)$ to a final point $B = (x_2, y_2)$, which is NOT located vertically under the starting point.

Which curve, $y(x)$, yields the shortest time of descent between the fixed points A and B?



$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2} \qquad \left(y' = \frac{dy}{dx}\right)$$

- The time of descent is

$$t_{AB} = \int \frac{ds}{v}$$

$ds \hat{=}$ small piece along path

- How to figure out speed v ?

A: From conservation of energy ($E_{pot} + E_{kin} = $ const.)

$$mgy = \frac{1}{2} m v^2 \Rightarrow v = \sqrt{2gy} \qquad (g = 9.8\ m s^{-2})$$

$$\Rightarrow \quad t_{AB} = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\, dx$$

- Define $\quad L = L(y, y') = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$

- Evaluate Euler-Lagrange equation (this takes a bit of work and patience, but is otherwise straightforward)

$\rightarrow$ Find: $\boxed{2yy'' + y'^2 + 1 = 0}$

Q: How to solve this one ?

- Trick: Use the following parametric form

$$\left.\begin{aligned} x &= a(\alpha - \sin\alpha) \\ y &= a(1 - \cos\alpha) \end{aligned}\right\} \quad \text{cycloid equations}$$

The parameter α is restricted to

$$0 \leq \alpha \leq \alpha_0 \leq 2\pi$$

- You have to fix a and α_0 so that curve $y(x)$ "hits" point $B = (x_2, y_2)$

→ These are the equations of (part of) a cycloid !

→ By plugging into equation, one shows that cycloid solves $2yy'' + y'^2 + 1 = 0$, and therefore is the desired brachistochrone solution !