Black Holes (cont'd)

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Note: something remarkable happens when $\frac{2GM}{c^2 r} = 1$

$$\Rightarrow R_s \equiv \frac{2GM}{c^2} = 3\text{ km}\left(\frac{M}{M_\odot}\right)$$

("Schwarzschild radius")

→ for photon emitted at $r = R_S$, $\nu_\infty = 0$,

no matter what ν_{em} is!

⇒ Photon ceases to exist!

→ Object becomes invisible!

⇒ BH

- Also: clocks at $r = R_s$ slow down to zero, as compared to clocks far away!

⇒ Time freezes at $r = R_s$ ("infinite time dilation")

→ Real meaning of R_s

- Structure of critical surface (or circumference) at $r = R_S$

→ Note: 2 pathological regions ("singularities") in SM

(i) $r = 0$; real, physical singularity
- GR breaks down, need quantum gravity

(ii) $r = R_s$; coordinate singularity (not real)
- Can be transformed away with suitable coordinates

→ To study geometry near $r = R_s$, choose

$$\bar{t} = t + \frac{R_s}{c} \ln \left| \frac{r}{R_s} - 1 \right| \quad \text{("Finkelstein coordinates")}$$

$$d\bar{t} = \frac{\partial \bar{t}}{\partial t} dt + \frac{\partial \bar{t}}{\partial r} dr$$

$$= dt + \frac{R_s}{cr} \frac{dr}{1 - R_s/r}$$

→ Eliminate the old time coordinate:

$$dt = d\bar{t} - \frac{\partial \bar{t}}{\partial r} dr$$

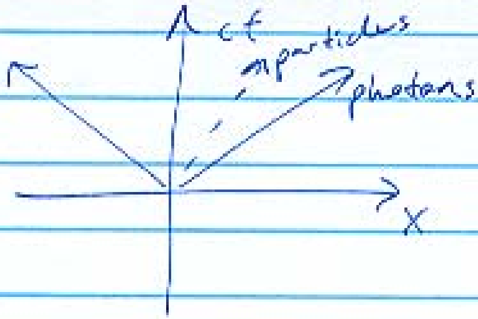
$$\Rightarrow ds^2 = -c^2\left(1 - \frac{R_s}{r}\right)d\bar{t}^2 + \frac{2cR_s}{r} d\bar{t}\, dr + \left(1 - \frac{R_s}{r}\right)dr^2 + r^2 d\Omega^2$$

Consider radial light rays $(d\varphi = d\theta = 0)$

$\Rightarrow d\Omega^2 = 0$

light rays $\Rightarrow$ Null geodesics $\Rightarrow ds^2 = 0$

$\rightarrow$ Consider light cones: e.g. in S.R.



$$\frac{dx}{dt} = \pm c.$$

Note: all particles with non-zero rest mass must remain inside light cone:

$$\frac{dx}{dt} = v < c$$

$\rightarrow$ Consider light cones in Schwarzschild geometry:

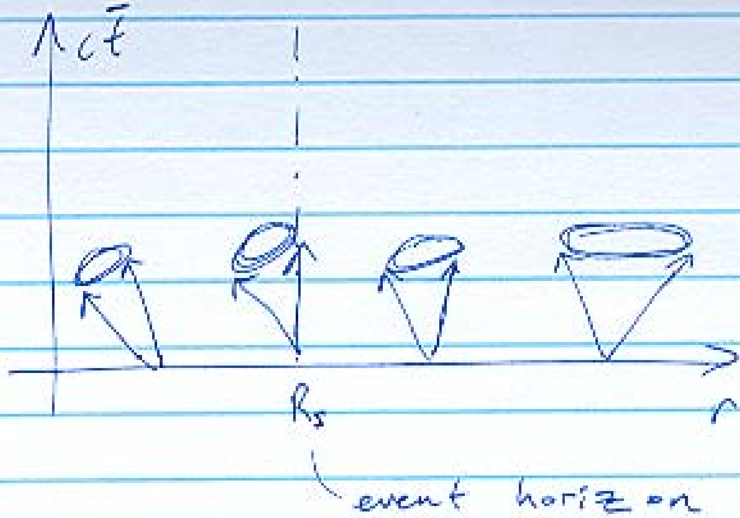
$$\left(1 - \frac{R_s}{r}\right)\left(\frac{dr}{dt}\right)^2 + 2c\frac{R_s}{r}\frac{dr}{dt} - c^2\left(1 - \frac{R_s}{r}\right) = 0$$

$\rightarrow$ quadratic equation for $dr/d\bar{t}$

$\rightarrow$ 2 solutions:

$$\left(\frac{dr}{d\bar{t}}\right)_1 = -c$$

$$\left(\frac{dr}{d\bar{t}}\right)_2 = c\,\frac{1 - R_s/r}{1 + R_s/r} \xrightarrow[r\to\infty]{} +c$$



→ Conjecture of cosmic censorship (Penrose 1968)

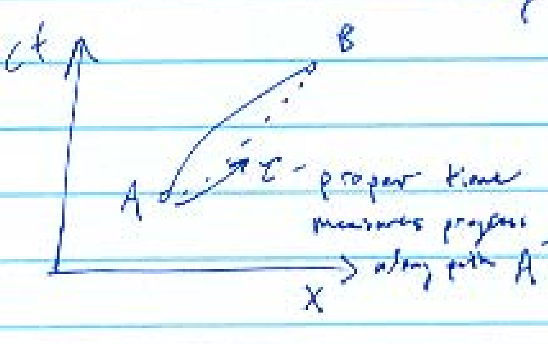
"Thou shalt not have naked singularities"

⇒ Every physical (real) singularity is cloaked inside event horizon!

• <u>Motion in Schwarzschild geometry</u>

→ revisit: geodesics

→ <u>Law of Motion</u>: particles move through spacetime along 'straightest' possible paths → geodesics



$$\int_A^B ds = \text{extremal}$$

$\Rightarrow$ Alternatively : $c \int_A^B d\tau = \text{extreme}$

("principle of extremal time")

$\Rightarrow$ trivially,

$$c \int_A^B \left(\frac{d\tau}{d\tau}\right) d\tau = \text{extreme}$$

– Consider SM in equatorial plane:

$$\left(\theta = \pi/2 \;;\; d\theta = 0\right)$$

$$c^2 d\tau^2 = -ds^2 = c^2\left(1 - \frac{R_s}{r}\right) dt^2 - \frac{dr^2}{1 - R_s/r} - r^2 d\varphi^2$$

$$\Rightarrow c^2 = c^2\left(1 - R_s/r\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - R_s/r} - r^2\dot{\varphi}^2$$

$$\dot{t} = \frac{dt}{d\tau} \;;\; \dot{r} = \frac{dr}{d\tau} \;;\; \dot{\varphi} = \frac{d\varphi}{d\tau}$$

$\rightarrow$ Define :

$$L = L\left(t, \dot{t}, r, \dot{r}, \varphi, \dot{\varphi}\right)$$

$$= \left[c^2\left(1 - \frac{R_s}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{\left(1 - R_s/r\right)} - r^2\dot{\varphi}^2\right]^{1/2}$$

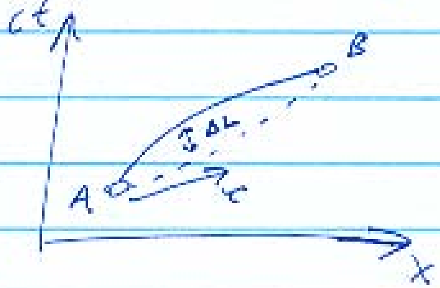
→ often called a "Lagrange function"

→ Note: $L = c$ always!

→ rephrase geodesic condition

$$\int_A^B L \, d\tau = \text{extreme}$$

→ Consider two neighboring paths



$$\int_A^B (L + \Delta L)\, d\tau - \int_A^B L \, d\tau = \int_A^B \Delta L \, d\tau = 0$$