Black Holes (cont'd)

→ $\Delta L = L(t+\Delta t, \dot{t}+\Delta\dot{t}, \dots) - L(t,\dot{t},\dots) =$

$$= \frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial \dot{t}}\Delta \dot{t} + \dots$$

– for extremum, need $\Delta L = 0$

$$\underbrace{\int_A^B \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial \dot{t}}\Delta \dot{t}\right) d\tau + \int_A^B \left(\frac{\partial L}{\partial r}\Delta r + \frac{\partial L}{\partial \dot{r}}\Delta \dot{r}\right) d\tau}_{\text{must be zero individually}} + \dots = 0$$

– notice: $\Delta\dot{t} = \frac{d}{d\tau}\Delta t$ etc

$$\int_A^B \left[\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial \dot{t}}\frac{d}{d\tau}\Delta t\right] d\tau = 0$$

$\Delta t = 0$ at $A + B$

$$\int_A^B \frac{\partial L}{\partial \dot{t}}\frac{d}{d\tau}\Delta t \, d\tau = \cancel{\left[\frac{\partial L}{\partial \dot{t}}\Delta t\right]_A^B}^{\,0} - \int_A^B \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right)\Delta t \, d\tau$$

integrate by parts

$$\Rightarrow \int_A^B \left[\frac{\partial L}{\partial t} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot t}\right)\right] \Delta t \, d\tau = 0 \quad \text{for all } \Delta t$$

$$\Rightarrow \boxed{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot t}\right) = \frac{\partial L}{\partial t}}$$

- by the same reasoning:

$$\boxed{\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot r}\right) &= \frac{\partial L}{\partial r} \\ \frac{d}{d\tau}\frac{\partial L}{\partial \dot\varphi} &= \frac{\partial L}{\partial \varphi} \end{aligned}}$$

"Euler - Lagrange equations"

- for SM: $\frac{\partial L}{\partial t} = \frac{\partial L}{\partial \varphi} = 0$ ($t + \varphi$ are "ignorable variables")

$\Rightarrow$ derive 2 constants of motion!

- Constants of Motion in Schwarzschild geometry

(1) $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot t}\right) = 0$

$$\frac{\partial L}{\partial \dot t} = \frac{c^2\left(1 - \frac{R_s}{r}\right)\dot t}{L}$$

- Thus $\frac{d}{d\tau}\left[c^2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau}\right] = 0$ [remember $L = c = $ const.]

$$\Rightarrow \boxed{c^2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau} = \text{const}}$$

$\rightarrow$ Q: Interpretation ?

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- recall particle energy in SR:

$$E = \gamma m_0 c^2$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \doteq \text{Lorentz factor}$$

- rephrase: $-c^2 d\tau^2 = ds^2 = -c^2 dt^2 + dx^2 = -c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right)$

$$(dx = v\,dt)$$

$$\Rightarrow \quad \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\Rightarrow \quad E = m_0 c^2 \frac{dt}{d\tau}$$

- Thus: $\boxed{e \equiv \frac{E}{m_0} = c^2\left(1 - \frac{R_S}{r}\right)\frac{dt}{d\tau} = \text{const}}$

("energy per unit rest mass")

→ note: recover SR form when $r \to \infty$, far away from mass M

(2) $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot\varphi}\right) = 0$

$$\frac{\partial L}{\partial \dot\varphi} = -\frac{r^2\dot\varphi}{L} \quad \Rightarrow \quad \frac{d}{d\tau}\left(r^2\dot\varphi\right) = 0$$

$$\Rightarrow \quad r^2\dot\varphi = \text{const}$$

- Q: interpretation l

- recall : angular momentum in Newton's theory

$$J = m_0 \, r \, v = m_0 r^2 \dot{\varphi}$$

$$(v = \omega r = \dot{\varphi} r)$$

- Thus:

$$\boxed{j \equiv \frac{J}{m_0} = r^2 \dot{\varphi} = \text{const}}$$

("angular momentum per unit rest mass")

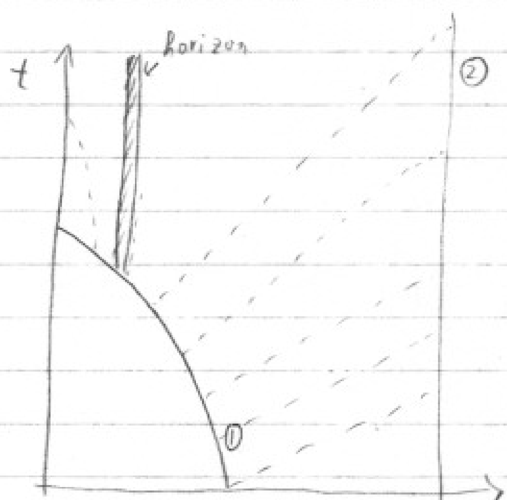
$\Rightarrow$ Motion in Schwarzschild geometry, s.t.:

$$\boxed{\begin{aligned} e &= \text{const} \\ j &= \text{const} \end{aligned}}$$

- Stellar Collapse

- Oppenheimer & Snyder (1939) : star with $M > M_{OV}$

$\rightarrow$ consider collapse from viewpoint of 2 observers:



① riding on surface of star

② far-away ($r \to \infty$)

- SM applicable for EXTERIOR of collapsing star, even though not time-independent $\rightarrow$ "Birkhoff's Theorem"

- Story told by observer ①:

→ "radial plunge" ($\dot{\varphi} = \dot{\vartheta} = 0$)

$$\Rightarrow \quad c^2 = c^2\left(1-\frac{R_s}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{R_s}{r}} \qquad \left(\dot{t} = \frac{dt}{d\tau}\,,\ \dot{r} = \frac{dr}{d\tau}\right)$$

(+)

$$e = c^2\left(1-\frac{R_s}{r}\right)\dot{t}$$

$$\Rightarrow \quad \boxed{\frac{dr}{d\tau} = -c\left[\frac{e^2}{c^4} - 1 + \frac{R_s}{r}\right]^{\frac{1}{2}}}$$

- $r(\tau)$ marks surface of collapsing star
- τ is proper time measured at collapsing surface

- Fix constant (e) such that plunge begins from rest at r_0:

$$\frac{e^2}{c^4} - 1 = -\frac{R_s}{r_0}$$

$$\Rightarrow \quad \frac{dr}{d\tau} = -c\left[\frac{R_s}{r} - \frac{R_s}{r_0}\right]^{\frac{1}{2}}$$

- Solve by separation of variables

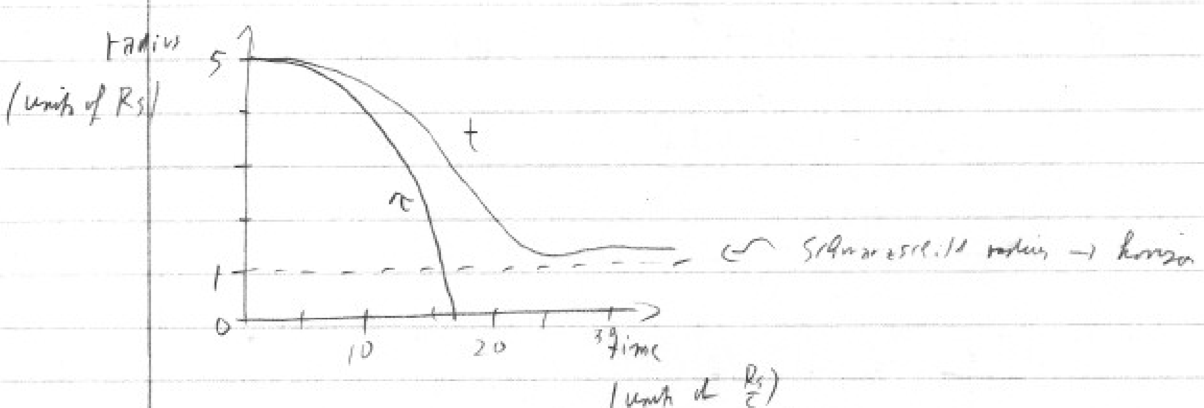
$$\sqrt{\frac{r_0}{R_s}}\,\frac{dr}{\sqrt{\frac{r_0}{r} - 1}} = -c\,d\tau$$

- introduce: $\eta = \arccos\left(2\frac{r}{r_0} - 1\right)$ ("cycloid parameter")

$$\frac{d\eta}{dr} = \frac{-1}{\sqrt{1-\left(2\frac{r}{r_0}-1\right)^2}}\,\frac{2}{r_0} = -\frac{1}{r}\,\frac{1}{\sqrt{\frac{r_0}{r}-1}}$$

- find:

$$\boxed{\tau(r) = \frac{1}{2}\frac{R_s}{c}\left(\frac{r_0}{R_s}\right)^{\frac{3}{2}} (\sin\eta + \eta)}$$



- $$\tau_0 = \tau(r=0) = \frac{\pi}{2}\frac{R_s}{c}\left(\frac{r_0}{R_s}\right)^{\frac{3}{2}} \qquad (\eta \to \pi)$$

  (e.g. $\tau_0 = 17.6 \frac{R_s}{c}$ for $r_0 = 5 R_s$)

- characteristic time for BH collapse:

$$\boxed{\tau_{BH} = \frac{R_s}{c} \simeq 10^{-5}\ s \left(\frac{M}{M_\odot}\right)}$$

→ Notice: – observer moving with collapsing surface does not notice anything unusual when crossing R_s
– will reach central singularity in finite time!

— Story told by observer ②:

→ again "radial plunge", but now we want $r = r(t)$, t = time measured by far-away observer

– 3 equations:

$$\left.\begin{array}{l}(i)\ \frac{dr}{d\tau} = -c\left[\frac{R_s}{r} - \frac{R_s}{r_0}\right]^{1/2} \\ (ii)\ e = c^2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau} \\ (iii)\ e = c^2\sqrt{1 - \frac{R_s}{r_0}}\end{array}\right\} \Rightarrow \boxed{\frac{dr}{dt} = -\frac{c}{\sqrt{\frac{r_0}{R_s} - 1}}\sqrt{\frac{r_0}{r} - 1}\left(1 - \frac{R_s}{r}\right)}$$

– Find approximate solution near horizon

$\Rightarrow$ for $r \to R_S$: $\frac{dr}{dt} \simeq -c\left(1 - \frac{R_S}{r}\right) = -\frac{c}{r}(r - R_S)$

$$\simeq -\frac{c}{R_S}(r - R_S)$$

$\Rightarrow$ separation of variables:

$$\frac{dr}{r - R_S} \simeq -\frac{c}{R_S}\,dt$$

– find: $\boxed{r(t) \simeq R_S + b \exp\left[-\frac{c}{R_S} t\right]}$

$\to$ Notice: – From viewpoint of far-away observer, radius approaches horizon only asymptotically

– will take infinite time $(t \to \infty)$ for $r = R_S$

– collapse appears to "freeze" at horizon

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- <u>More General Black Holes</u>

  - BH fully defined by 3 parameters: M, J, Q (electric charge)

  - all other properties are "forgotten"

    → $\boxed{\text{"BHs have no hair"}}$

<u>E.g.:</u>