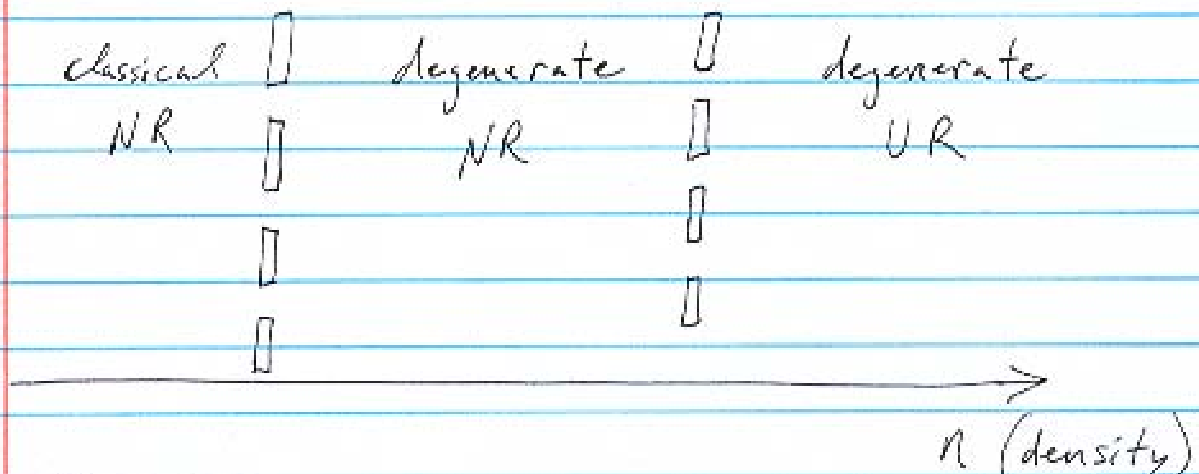
Jan. 24, 2008

Basic Physics of Compact Objects (cont'd)

- **Classical / quantum boundary**

- preview:



- Q: When does quantum mechanics (QM) become important?

- define: average distance between particles

$$n = \frac{N}{V} = \frac{1}{\ell^3} \Rightarrow \boxed{\ell \simeq n^{-1/3}}$$

- consider QM waves associated with particle ("wave/particle duality)

$$\lambda_{deB} = \frac{h}{p} \quad (\text{"de Broglie wavelength"})$$

$$h \hat{=} \text{Planck's constant}$$

- Now: QM effects kick in when

$$l \lesssim \lambda_{deB}$$

$$n^{-1/3} \lesssim \frac{h}{p} \Rightarrow p \lesssim h n^{1/3}$$

- slow-moving + dense gas is in danger of becoming degenerate

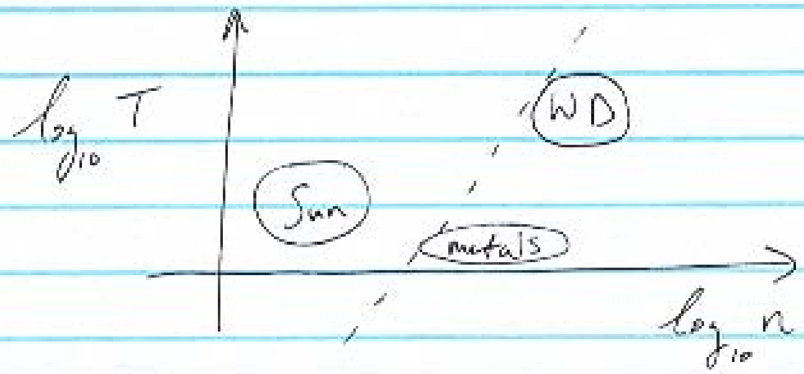
- re-phrase: slow-moving $\leftrightarrow$ cold

$$\frac{p^2}{2m_0} = \frac{3}{2} k_B T$$

$$\frac{h^2 n^{2/3}}{2m_0} = \frac{p^2}{2m_0} \simeq k_B T$$

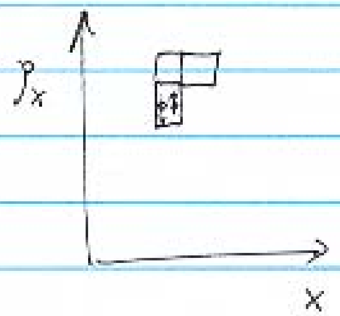
$\Rightarrow$ Find classical / quantum boundary!

$$\boxed{T \simeq \text{constant} \times n^{2/3}}$$



• Quantum ("degenerate") gas

– consider 6-dimensional phase-space (x, y, z, p_x, p_y, p_z)



quantum cells: $\underbrace{dV}_{dx\,dy\,dz}\ \underbrace{d^3p}_{dp_x\,dp_y\,dp_z} \simeq h^3$

(recall Heisenberg's uncertainty principle)

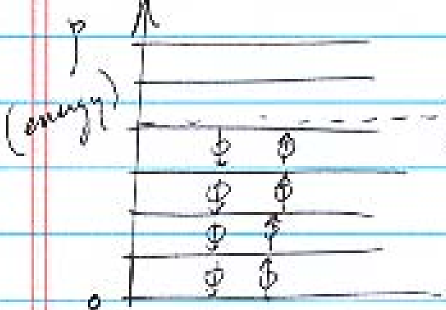
$$\boxed{dx\, dp_x \gtrsim h}$$

Pauli's exclusion principle

"2 or less fermions (e.g. electrons) per quantum cell"

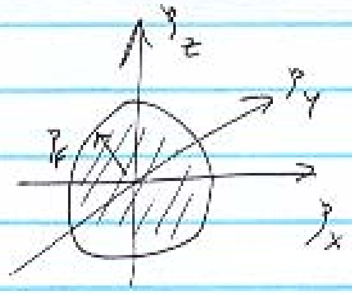
– consider extreme case of tight packing ("complete degeneracy")

⟶ All fermions have lowest possible energy (or momentum)



p_F = "Fermi momentum" ("surface of the Fermi sea")

- Visualize 3D momentum part of 6D phase-space (for complete degeneracy)



volume $= \frac{4\pi}{3} p_F^3$

(a sphere → no preferred direction → "isotropic" distribution)

- Q: How many particles can we fit into maximally dense phase space?

- A: $N = 2 \dfrac{\frac{4\pi}{3} p_F^3 V}{h^3} = \dfrac{8\pi}{3h^3} p_F^3 V$

$$\Rightarrow n = \frac{8\pi}{3h^3} p_F^3 \Rightarrow \boxed{p_F = \left(\frac{3}{8\pi}\right)^{1/3} h\, n^{1/3}}$$

<u>Case 2</u>: quantum + NR

- gas is NR $\Rightarrow P = \frac{2}{3} u_{kin}$

- estimate u_{kin} :

$$u_{kin} = n \langle \epsilon_{kin} \rangle$$

$$\left.\begin{array}{l} \langle \epsilon_{kin} \rangle \simeq \dfrac{p_F^2}{2m_0} \simeq \dfrac{p_F^2}{m_0} \\ \\ n \simeq \dfrac{p_F^3}{h^3} \end{array}\right\} u_{kin} = \frac{p_F^5}{h^3 m_0}$$

$\rightarrow$ exact result: $u_{kin} = \dfrac{8\pi}{10} \dfrac{p_F^5}{h^3 m_0}$

$$\Rightarrow P = \frac{2}{3} u_{kin} = \underbrace{\frac{h^2}{5 m_0} \left(\frac{3}{8\pi}\right)^{2/3}}_{K_{NR}} n^{5/3}$$

$$\Rightarrow \boxed{P = K_{NR}\, n^{5/3}}$$

Case 3: quantum gas + UR

- gas is UR $\Rightarrow P = \frac{1}{3} u_{kin}$

$\rightarrow u_{kin} = n \langle \epsilon_{kin} \rangle$

$\langle \epsilon_{kin} \rangle \simeq p_F c$

$$n = \frac{p_F^3}{h^3}$$

With these, $U_{kin} \simeq \frac{c}{h^3} p_F^4$

exact result: $U_{kin} = \frac{2\pi c}{h^3} p_F^4$

$$\Rightarrow P = \frac{1}{3} U_{kin}$$

$$= \underbrace{\frac{hc}{4} \frac{3}{8\pi}}_{K_{UR}} n^{4/3}$$

$$\Rightarrow \boxed{P = K_{UR} n^{4/3}}.$$

Quantum NR / UR boundary

$v \simeq c$

$$\Rightarrow p_F = m_0 v \simeq m_0 c$$

But $p_f \simeq h n^{1/3}$

$$n_{UR} \simeq \left(\frac{c m_0}{h}\right)^3$$

For electrons, $m_0 = m_e = 9.1 \times 10^{-31}$ kg

$\Rightarrow n_{UR} \simeq 10^{36}\ m^{-3}$

$\Rightarrow \rho_{UR} \simeq m_H\, n_{UR} \simeq 10^{9}\ kg\, m^{-3}$