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Basic Physics of Compact Objects (cont'd)

Pressure of an ideal gas

- Non-relativistic particles (NR)

$v \ll c \qquad \epsilon_{kin} \ll m_o c^2$

rest mass

$$\left.\begin{aligned} p &= mv \\ \epsilon_{kin} &= \frac{1}{2} m v^2 \end{aligned}\right\} \quad pv = 2\epsilon_{kin}$$

With this, the pressure is, for the NR case,

$$\boxed{P = \frac{1}{3}\frac{N}{V}\langle \vec{p}\cdot\vec{v}\rangle = \frac{2}{3}\frac{N}{V}\langle \epsilon_{kin}\rangle \equiv \frac{2}{3} U_{kin}},$$

where $U_{kin} \equiv$ kinetic energy density

- Ultra-relativistic particles (UR)

$$\left.\begin{aligned} &\epsilon_{kin} \gg m_o c^2 \qquad v \simeq c \\ &\epsilon_{kin} = pc \end{aligned}\right\} \quad pv = \epsilon_{kin}$$

$$\Rightarrow \boxed{P = \frac{1}{3}\,\frac{N}{V}\,\langle \vec{p}\cdot\vec{v}\rangle = \frac{1}{3}\,U_{kin}} \quad (UR)$$

Total Energy and Stability

(1) Consider NR gas!

- What is needed: $\langle P \rangle = -\frac{1}{3}\,\frac{E_{grav}}{V}$

- What is available: We have kinetic energy because of the random motion of the particles

$$\langle P \rangle = \frac{2}{3}\,\langle U_{kin}\rangle = \frac{2}{3}\,\frac{E_{kin}}{V}$$

$E_{kin} \mathrel{\hat{=}}$ total kinetic energy of the star

- For a star in hydrostatic equilibrium

$$\frac{2}{3}\,\frac{E_{kin}}{V} = -\frac{1}{3}\,\frac{E_{pot}}{V}$$

$$\Rightarrow \boxed{2E_{kin} = -E_{pot}}$$ The virial theorem. (VT)

Total energy:

$$E_{tot} = E_{kin} + E_{pot}$$

System will be gravitationally bound, if

$$E_{tot} < 0$$

(2) Consider UR gas

- What we need: $\langle P \rangle = -\frac{1}{3} \frac{E_{grav}}{V}$
- What we have: $\langle P \rangle = \frac{1}{3} U_{kin} = \frac{1}{3} \frac{E_{kin}}{V}$

$\Longrightarrow$ $E_{tot} = E_{kin} + E_{pot} = 0$

$\Longrightarrow$ UR stars are unstable!

Pressure inside a star

- Known: $\rho(r)$, T

$\rightarrow$ Calculate $P = P(\rho, T, \ldots)$

("equation of state"; e.o.s.)

- Possible combinations

classical ("non-degenerate")	quantum ("degenerate")
non-relativistic	ultra-relativistic
$\epsilon_{kin} \ll m_0 c^2$	$\epsilon_{kin} \gg m_0 c^2$

Case 1: classical + NR

→ "normal" gas (e.g. air in classroom)

→ gas is NR: $\Rightarrow P = \frac{2}{3} U_{kin}$

- Q: What is U_{kin}?

- A: $U_{kin} = n \langle \epsilon_{kin} \rangle$

→ From kinetic theory

Three degrees of freedom for this gas: $\langle \epsilon_{kin} \rangle = \frac{3}{2} k_B T$

$$\langle \epsilon_{kin} \rangle = 3 \times \frac{1}{2} k_B T \text{ , where}$$

$k_B \hat{=}$ Boltzmann's constant $= 1.38 \times 10^{-23}\ J K^{-1}$

$$\Longrightarrow \boxed{P = n k_B T} \quad (\text{"ideal gas law"})$$

$\rightarrow$ This holds for the gas in the sun.