Jan. 17, 2008

Physics of Compact Objects

"A theorist's star"

eg. Sun

$M_\odot = 2 \times 10^{30}$ kg

$R_\odot = 6.9 \times 10^{8}$ m

- perfect sphere
- no rotation
- no magnet fields

...

- average density: $\langle \rho \rangle = \dfrac{M}{\frac{4\pi}{3} R^3}$

eg. $\langle \rho \rangle_\odot = 1400$ kg m^{-3}

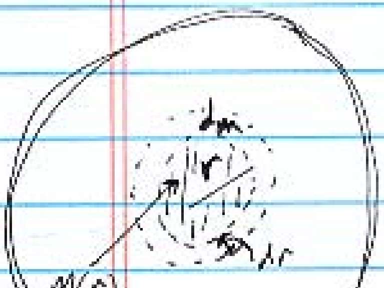
- Mechanical Structure

→ Equation of motion (Newton's 2nd Law)

total force = mass × acceleration

Figure out mass → $dm = 4\pi r^2 \rho(r)$ (mass shell)

$$m(r) = 4\pi \int_0^r r'^2 \rho(r')\, dr'$$

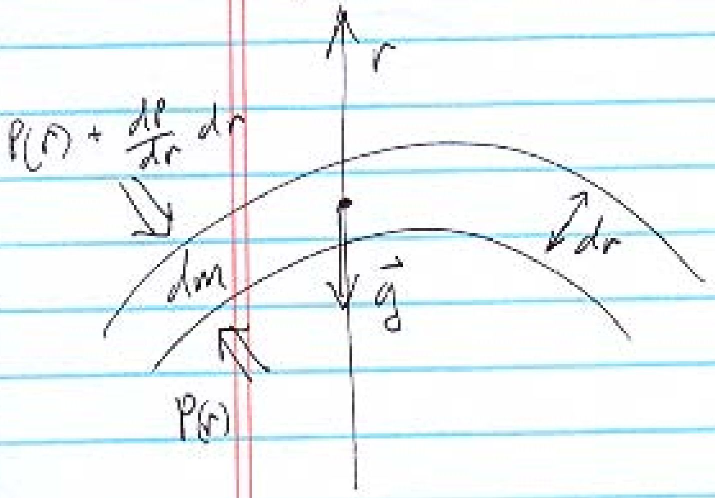


$dm \hat{=}$ mass in a shell of thickness dr

$m(r) \hat{=}$ mass enclosed by the shell, with radius r

Figure out total force $\longrightarrow$ (force due to gravity) $= -g(r)\,dm$

$$= -\frac{G M(r)}{r^2}\,dm$$



(force due to pressure) $= 4\pi r^2\, P(r)$

$$- 4\pi r^2 \left[P(r) + \frac{dP}{dr}\,dr\right]$$

$$= -\frac{dP}{dr}\,4\pi r^2\,dr$$

$$= \frac{-1}{\rho}\frac{dP}{dr}\,\underbrace{4\pi r^2 \rho\, dr}_{dm}$$

$$= \frac{-1}{\rho}\frac{dP}{dr}\,dm$$

Total force $= -g\,dm - \frac{1}{\rho}\frac{dP}{dr}\,dm$

$$= \left[-g - \frac{1}{\rho}\frac{dP}{dr}\right] dm$$

mass $\times$ acceleration $= dm \times \frac{d^2 r}{dt^2}$

$\Longrightarrow$ $\boxed{\frac{d^2 r}{dt^2} = -g - \frac{1}{\rho}\frac{dP}{dr}}$ is our equation of motion.

Thought experiment $\longrightarrow$ switch off pressure term!

$$\frac{d^2 r}{dt^2} \simeq \frac{-R}{\tau_{ff}^2} = -g \simeq \frac{-GM}{R^2}$$

$$\Rightarrow \boxed{\tau_{ff} \simeq \frac{1}{\sqrt{G\frac{M}{R^3}}} \simeq \frac{1}{\sqrt{G\rho}}}$$ (free-fall timescale)

e.g. For the Sun, $\tau_{ff} \sim 1$ hour

But the Sun has not changed for a very long time!

$\Rightarrow$ Pressure $\simeq$ gravity

$\rightarrow$ Hydrostatic equilibrium $\rightarrow \frac{d^2r}{dt^2} = 0$

$$\Rightarrow \boxed{\frac{dP}{dr} = -\rho g}$$

- Simple estimate for central pressure P

$$\frac{dP}{dr} \simeq \frac{P_0 - P_c}{R - 0} \simeq \frac{-P_c}{R}$$, as we can neglect the small pressure P_0 at the star's surface

$$-\rho g \simeq \frac{-M}{R^3}\frac{GM}{R^2} = -\frac{GM^2}{R^5}$$

$$\Rightarrow P_c \simeq \frac{GM^2}{R^4}$$, where we also use the condition of hydrostatic equilibrium

e.g. Sun $P_{c,\odot} \simeq 10^{15}$ Pa, and Earth's atmospheric pressure is $\simeq 10^5$ Pa

- A different way to look at hydrostatic equilibrium → Describe the strength of gravity using the gravitational potential energy

$$E_{pot} = -\int_0^R \frac{G\,M(r)\,dm}{r}$$

→ What is the average pressure needed to support the star against gravity?

- Calculate average pressure:

$$\langle P \rangle = \frac{\int_0^R 4\pi r'^2 P(r')\,dr'}{\int_0^R 4\pi r'^2\,dr'}$$

- Use integration by parts:

$$\int_0^R 4\pi r'^2 P(r')\,dr' = \cancelto{0}{\frac{4\pi}{3}\Big[P r^3\Big]_0^R} - \frac{1}{3}\int_0^R 4\pi r^3 \frac{dP}{dr}\,dr$$

→ Use equation of hydrostatic equilibrium:

$$4\pi r^3 \frac{dP}{dr}\,dr = -4\pi r^3 \rho \frac{GM}{r^2}\,dr = -4\pi r^2 \rho \frac{GM}{r}\,dr$$

$$= \frac{-GM}{r}\, dm$$

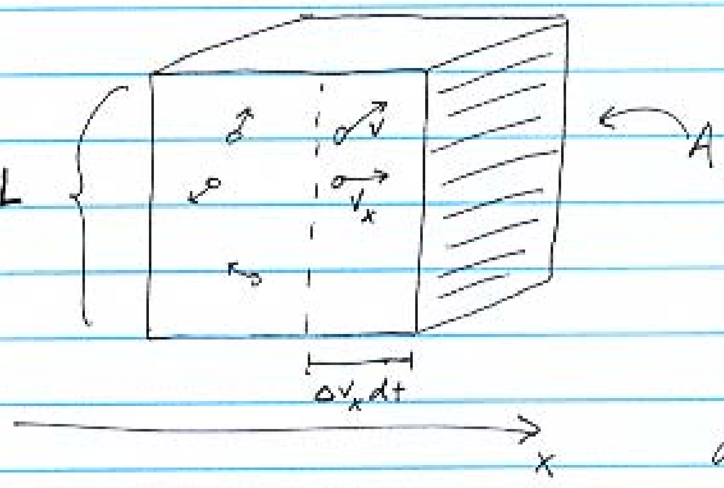
With this, we now have

$$\boxed{\int_0^R 4\pi r^2 P(r)\, dr = \frac{-1}{3}\int_0^R \frac{GM}{r}\, dm = -\frac{1}{3} E_{pot}}$$

$$\Rightarrow \boxed{\langle P \rangle = \frac{-1}{3}\frac{E_{pot}}{V}}$$, where V is the volume of the star.

• What is pressure?

$$P = \frac{\text{Force}}{\text{Area}} = \frac{dp}{A\, dt}$$ ("Momentum flux")



$$V = L^3$$

number density

$$N = n L^3$$

$$dV = A v_x dt$$

$$dN = \frac{1}{2} n\, dV = \frac{1}{2} n v_x A dt$$

Half of the particles are moving away from the wall of the box

$$dp_x = 2\, dN_{p_x} = n\, A\, p_x\, v_x\, dt$$

$$\Longrightarrow P = \frac{dp_x}{A\, dt} = n \langle p_x v_x \rangle$$

$$\langle \vec{p} \cdot \vec{v} \rangle = \langle p_x v_x \rangle + \langle p_y v_y \rangle + \langle p_z v_z \rangle$$

$$= 3 \langle p_x v_x \rangle \quad (\text{"isotropic pressure"})$$

$$\Longrightarrow \boxed{P = \frac{n}{3} \langle \vec{p} \cdot \vec{v} \rangle}$$