Homework #4

$\boxed{1}$ a) The Schwarzschild radius of an object scales linearly with its mass, and a $1 M_\odot$ star has a Schwarzschild radius of 3 km. Therefore, here we have

$$R_s = 3 \times 10^6 \text{ km} \quad \text{for the BH.}$$

This distance is also 10 light-seconds.

The time it takes to fall to the singularity is

$$\tau_0 = \frac{\pi}{2} \frac{R_s}{c}$$

$$\simeq 17 \text{ sec.}$$

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b) For a purely radial orientation, $d\varphi = d\theta = 0$, and here we consider $dt = 0$ too. Then, the proper length of the stick is

$$(1m)^2 = \frac{(\Delta r)^2}{\left(1 - \frac{R_s}{r}\right)}$$

With $r = 10^2 R_s$, we find that the coordinate distance taken up is

$$\Delta r = (0.99)^{1/2} \, m$$

$$= 0.995 \, m$$

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c) The area of the shell is given by $4\pi r^2$, where r is the coordinate distance from the singularity. Thus, the shell takes up the space between the coordinate distances

$$r_1 = 3R_s \qquad \text{and} \qquad r_2 = 5 R_s ,$$

where $R_s = \frac{2GM}{c^2}$.

The physical distance is given by

$$r_{phys} = \int_{3R_s}^{5R_s} \frac{dr}{\sqrt{1 - R_s/r}} ,$$

which yields

$$r_{phys} = \sqrt{r(r-R_s)} + R_s \ln\left[\sqrt{r} + \sqrt{r - R_s}\right]\Bigg|_{3R_s}^{5R_s}$$

$$= \sqrt{20}\, R_s - \sqrt{6}\, R_s + R_s \ln\left[\sqrt{5 R_s} + \sqrt{4 R_s}\right] - R_s \ln\left[\sqrt{3R_s} + \sqrt{2R_s}\right]$$

$$= \left(\sqrt{20} - \sqrt{6} + \ln\left(\frac{\sqrt{5} + \sqrt{4}}{\sqrt{3} + \sqrt{2}}\right)\right) R_s$$

$$= \left(2.02 + \ln\left(\frac{4.236}{3.146}\right)\right) R_s$$

$$= 2.317\, R_s\,.$$

$\boxed{2}$ a) Because there is no dependence on t or ϕ in the Lagrange function L,

we have

(1) $$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{\phi}} \right) = 0$$ and

(2) $$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{t}} \right) = 0 .$$

Thus, the expressions in parentheses in the above two equations are constants in time τ. From equation (1), we have

$$r^2 \dot{\phi} = \text{constant},$$

and from equation (2),

$$\dot{t} = \text{constant}.$$

The first of these is the conservation of angular momentum, while the second is the conservation of energy.

b) Applying the Euler - Lagrange equation gives

$$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\Rightarrow \quad -2r\dot{\phi}^2 = -2\ddot{r}$$

$$\Rightarrow \quad \ddot{r} = r\dot{\phi}^2 .$$

The force is the centripetal force.