HW #3

1 a) To begin, expand the square root term as follows:

$$\sqrt{1-\left(R_s/R\right)} \simeq 1 - \frac{R_s}{2R},$$

for $R \gg R_s$. With this approximation,

$$P_c = \rho_0 c^2 \frac{1-\sqrt{1-R_s/R}}{3\sqrt{1-R_s/R}-1}$$

$$\simeq \frac{\rho_0 c^2 \left(R_s/R\right)}{2\left(2-\frac{3}{2}\left(\frac{R_s}{R}\right)\right)}$$

$$\simeq \frac{\rho_0 c^2 R_s}{4R}$$

$$= \frac{\rho_0 G M}{2R}$$

$$= \frac{2\pi}{3} G \rho_0^2 R^2, \text{ where } R_s = \frac{2GM}{c^2}.$$

b) With $\frac{R_s}{R} = 0.5$, we obtain

$$P_c = \frac{0.29}{1.12} \rho_0 c^2$$

$$= 0.25 \rho_0 c^2$$

---

c) We have

$$P_c = \frac{hc}{4} \left(\frac{3}{8\pi m_H^4} \right)^{1/3} \rho_0^{4/3}$$

$$= 1.1 \times 10^{10} \rho_0^{4/3} \quad \text{(Pascal)}$$

Equating the two pressures gives

$$2.25 \times 10^{16} \rho_0 = 1.1 \times 10^{10} \rho_0^{4/3}$$

$$\Rightarrow \boxed{\rho_0 = 9 \times 10^{18} \ \text{kg m}^{-3}}$$

d) Using the fact that $\frac{R_s}{R} = 0.5$, we have

$$M = \frac{4\pi}{3} R^3 \rho_0$$

$$= \frac{4\pi}{3} \left(\frac{4GM}{c^2}\right)^3 \rho_0$$

$$\Rightarrow \quad M = \left(\frac{3c^6}{256\pi G^3 \rho_0}\right)^{1/2}$$

$$= 0.5 \, M_\odot$$

With this, we have

$$R = \left(\frac{3M}{4\pi \rho_0}\right)^{1/3}$$

$$= 3 \text{ km}$$

$\boxed{2}$ a) We have

$$z \equiv \frac{\lambda_\infty - \lambda_0}{\lambda_0}$$

$$= \frac{\nu_0}{\nu_\infty} - 1$$

$$= \frac{t_\infty}{t_0} - 1 ,$$

where ν_0 and t_0 are the rest-frame frequency and corresponding period of the emitted photon. Gravitational time dilation gives

$$\frac{t_\infty}{t_0} = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} .$$

Thus,

$$z = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} - 1 = 0.00015$$

b) Because the laser and the observer are falling together, they are essentially in the same reference frame and so the observer will always measure 632.8 nm.