Problem Set #1

① (a) We have
$$M = \int_0^R 4\pi r^2 \rho(r)\,dr$$
$$= 4\pi\rho_c \int_0^R r^2\left[1 - \left(\frac{r}{R}\right)^2\right]dr$$
$$= 4\pi\rho_c\left[\frac{R^3}{3} - \frac{R^3}{5}\right]$$
$$\Rightarrow \quad M = \frac{8\pi\rho_c}{15}R^3$$
$$\approx \boxed{220\ M_\odot},$$

using $\rho_c = 3\times10^4\ \mathrm{kg\ m^{-3}}$ and $R = 3R_\odot$.

Note: $M_\odot = 2\times10^{30}\ \mathrm{kg}$

$R_\odot = 6.9\times10^8\ \mathrm{m}$.

(b) The average density is simply the total mass of the star divided by its total volume:

$$\langle \rho \rangle = \frac{M}{\left(\frac{4\pi}{3} R^3\right)}$$

$$= \boxed{1.2 \times 10^4 \ \mathrm{kg\ m^{-3}}}.$$

Thus, the free-fall time is

$$\tau_{ff} \simeq \frac{1}{\sqrt{G \langle \rho \rangle}}$$

$$\simeq 1{,}000 \ \mathrm{sec} \ (\sim 18 \ \mathrm{min}).$$

This is the time required for the star to collapse under solely the force of its own gravity.

(c) We have the equation for hydrostatic equilibrium as follows:

$$\frac{dP}{dr} = -\rho g = -\rho(r)\,\frac{G\,M(r)}{r^2}.$$

We must first express the mass as a function of radius, $M(r)$:

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\,dr'$$

$$= 4\pi\rho_c\left[\left.\frac{r'^3}{3}\right|_0^r - \left.\frac{r'^5}{5R^2}\right|_0^r\right]$$

$$= 4\pi\rho_c\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right].$$

With this, we arrive at

$$\frac{dP}{dr} = -\frac{4\pi G\rho_c^2}{r^2}\left[1-\left(\frac{r}{R}\right)^2\right]\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right],$$

or, more simply,

$$\frac{dP}{dr} = -4\pi G \rho_c^2 \left[\frac{r}{3} - \frac{r^3}{3R^2} - \frac{r^3}{5R^2} + \frac{r^5}{5R^4} \right]$$

$$= -4\pi G \rho_c^2 \left[\frac{r}{3} + \frac{r^5}{5R^4} - \frac{8r^3}{15R^2} \right].$$

Integrating yields

$$P(r) = \int_0^P dP' = \int_R^r -4\pi G \rho_c^2 \left[\frac{r'}{3} + \frac{r'^5}{5R^4} - \frac{8r'^3}{15R^2} \right] dr'$$

$$= -4\pi G \rho_c^2 \left[\frac{r'^2}{6} + \frac{r'^6}{30R^4} - \frac{2r'^4}{15R^2} \right] \Bigg|_R^r$$

$$= 4\pi G \rho_c^2 \left[\frac{R^2}{6} + \frac{R^2}{30} - \frac{2R^2}{15} - \frac{r^2}{6} - \frac{r^6}{30R^4} + \frac{2r^4}{15R^2} \right]$$

$$= \frac{8\pi G \rho_c^2 R^2}{15}\left[1 - \frac{5}{4}\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4 - \frac{1}{4}\left(\frac{r}{R}\right)^6\right].$$

Thus,

$$K = \frac{8\pi G \rho_c^2 R^2}{15} \quad \text{and}$$

$$a_1 = 0$$

$$a_2 = -5/4$$

$$a_3 = 0$$

$$a_4 = 1$$

$$a_5 = 0$$

$$a_6 = -1/4,$$

with all other $a_i = 0$.

(d) The pressure at the center of the star is

$$P_c = P(r=0) = \frac{8\pi G \rho_c^2 R^2}{15}$$

$$= \boxed{5 \times 10^{16} \text{ Pa}}$$

From class, we have the approximation

$$P_c = \frac{GM^2}{R^4}$$

$$= \boxed{8 \times 10^{16} \text{ Pa}},$$ less than a factor of two off from the exact value.

(e) In general for a spherical star,

$$E_{grav} = \int_0^M \frac{-GM}{r}\, dm$$

$$= -4\pi G \int_0^R \frac{M(r)\,\rho(r)}{r} r^2 dr$$

$$= -4\pi G \int_0^R r\left[4\pi \rho_c \left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]\right]$$

$$\times \rho_c \left[1 - \left(\frac{r}{R}\right)^2\right] dr$$

$$= -(4\pi)^2 G \rho_c^2 \int_0^R \left(\frac{r^4}{3} - \frac{r^6}{5R^2}\right)\left[1 - \left(\frac{r}{R}\right)^2\right] dr$$

$$= -(4\pi)^2 G \rho_c^2 \int_0^R \left(\frac{r^4}{3} - \frac{r^6}{5R^2} - \frac{r^6}{3R^2} + \frac{r^8}{5R^4}\right) dr$$

②

$$= -(4\pi\rho_c)^2 G\left[\frac{R^5}{15} - \frac{R^5}{35} - \frac{R^5}{21} + \frac{R^5}{45}\right]$$

$$= -(4\pi)^2 G\left[\frac{15M}{8\pi R^3}\right]^2 R^5\left[\frac{1}{15} - \frac{1}{35} - \frac{1}{21} + \frac{1}{45}\right]$$

$$= \frac{-GM^2}{4R}\left[15 - \frac{45}{7} - \frac{75}{7} + 5\right]$$

$$= \frac{-GM^2}{28R}\left[140 - 45 - 75\right]$$

$$= \boxed{-\left(\frac{5}{7}\right)\frac{GM^2}{R}}$$

For our star, $\boxed{E_{grav.} = -2 \times 10^{43}\ \text{J}}$

② (a) Squaring the energy equation on both sides, we have

$$\epsilon^2 = \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}}$$, and dividing

the momentum by the energy, we have

$$\frac{p}{\epsilon} = \frac{v}{c^2} \Rightarrow \left(\frac{pc}{\epsilon}\right)^2 = \frac{v^2}{c^2}.$$

Substituting this expression for $\left(\frac{v}{c}\right)^2$ into the equation for ϵ^2 above, we obtain

$$\epsilon^2 = \frac{m_0^2 c^4}{1 - \frac{p^2 c^2}{\epsilon^2}}$$

$$\Rightarrow \boxed{m_0^2 c^4 = \epsilon^2 - p^2 c^2.}$$ This equation tells us that a particle has a rest energy $m_0 c^2$, in addition to its energy due to momentum.

(b) First, expand the energy in $\left(\frac{v}{c}\right)^2$, using the binomial theorem:

$$\epsilon = m_o c^2 \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2}$$

$$\simeq m_o c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)$$

$$\simeq m_o c^2 + \frac{1}{2} m_o v^2, \quad \text{for} \quad v << c.$$

Thus, $\epsilon - m_o c^2 = \epsilon_{kin}$

$\simeq \frac{1}{2} m_o v^2$, for

slow-moving (NR) particles.

For $v \simeq c$, we have

$$\epsilon_{kin} = \epsilon - m_o c^2$$

$$= \left((m_o c^2)^2 + p^2 c^2\right)^{1/2} - m_o c^2$$

$$= pc\left(1 + \frac{m_o^2 c^4}{p^2 c^2}\right)^{1/2} - m_o c^2$$

$$= pc\left(1 + \frac{c^2\left(1 - \frac{v^2}{c^2}\right)}{v^2}\right) - m_o c^2,$$

using $p = \dfrac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$. Now, plugging in $v = c$, we obtain

$$\boxed{\epsilon_{kin} = pc - m_o c^2.}$$

Now, note that the ratio of pc to $m_o c^2$, for

$V \simeq c$, is

$$\frac{pc}{mc^2} = \frac{v}{c}\left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2}$$

and this ratio goes to ∞ as $v \to c$. Thus, for $v \simeq c$, $pc \gg mc^2$ and we can throw out the mc^2 squared term in

$$E_{kin} \Rightarrow \boxed{E_{kin} \simeq pc.}$$

(3) The virial theorem gives

$$\langle P \rangle = -\frac{1}{3}\frac{E_{grav}}{V}.$$ For this case,

$$E_{grav} = -\frac{GM^2}{R}$$ and $$V = \frac{M}{\langle \rho \rangle}.$$

Using also $$R = \left(\frac{3M}{4\pi \langle \rho \rangle}\right)^{1/3},$$ we have

$$\langle P \rangle = \frac{1}{3} G \frac{M^2 \langle \rho \rangle}{\left(\frac{3M}{4\pi \langle \rho \rangle}\right)^{1/3} M}$$

$$= G \left(\frac{4\pi}{3^4}\right)^{1/3} M^{2/3} \langle \rho \rangle^{4/3}.$$

Now, we also have

$$\langle P \rangle = n k_B T$$

$$= \frac{\langle \rho \rangle}{\bar{m}} k_B T,$$ where $$\bar{m} = \frac{m_p + m_e}{2}$$

$$\simeq \frac{m_p}{2},$$

for hydrogen gas (ionized!). Thus,

$$2\,\frac{\langle\rho\rangle k_B T}{m_p} = G\left(\frac{4\pi}{3}\right)^{1/3} M^{2/3}\langle\rho\rangle^{4/3}.$$

$$\Rightarrow M^{2/3} = \frac{2}{m_p}\left(\frac{3}{4\pi}\right)^{1/3}\frac{k_B T}{G}\langle\rho\rangle^{-1/3}$$

$$\Rightarrow M = \left(\frac{2k_BT}{G\,m_p}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\langle\rho\rangle^{-1/2}$$

With $\langle\rho\rangle = \langle\rho_\odot\rangle = 1.4\ \mathrm{g\ cm^{-3}}$ and $T = 10^6\ \mathrm{K}$,

$$\boxed{M = 0.13\,M_\odot.}$$