Feb. 21, 2008

General Relativity (cont'd)

• Relativistic Field Equation

- Recall Newton: $\nabla^2 \varphi = 4\pi G \rho$

- Newtonian spacetime - $g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right)$

$$\Rightarrow \nabla^2 g_{00} = \frac{-2}{c^2} \nabla^2 \varphi$$

$$= -\frac{8\pi G}{c^2} \rho$$

- Q: What are the sources of gravity?
  - Newton: ρ (mass density)
  - Einstein: $\left\{ \begin{array}{l} \rho c^2 \text{ (mass-energy density)} \\ \text{pressure } p \text{ (has units of } \frac{\text{energy}}{\text{volume}}\text{)} \end{array} \right.$

$\longrightarrow$ Neatly organize sources of gravity

$$T_{\mu\nu} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \end{array} \quad \text{("Stress-energy tensor")}$$

→ total source of gravity, "effective density"

$$\boxed{\begin{aligned}\rho_{eff} &= \frac{1}{c^2}\left(T_{00} + T_{11} + T_{22} + T_{33}\right)\\ &= \rho + \frac{3p}{c^2}\end{aligned}}$$

→ with $T_{00} = \rho c^2$

$$\Rightarrow \nabla^2 g_{00} = \frac{-8\pi G}{c^4} T_{00}$$

in general, replace $\nabla^2 g_{00} \longrightarrow G_{\mu\nu}$

$$\underbrace{G_{\mu\nu}}_{\text{'Einstein tensor'}} = f\left[2^{nd} \text{ derivatives of } g_{\mu\nu},\ 1^{st} \text{ derivatives of } g_{\mu\nu},\ g_{\mu\nu}\right]$$

"Matter tells space how to curve"

$$G_{\mu\nu} = \frac{-8\pi G}{c^4} T_{\mu\nu}$$

⇒ schematically : $\left(\text{curvature of spacetime}\right) = \frac{-8\pi G}{c^4}\left(\text{energy density}\right)$

→ VERY complicated!

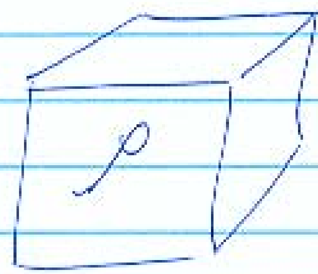
10 coupled non-linear PDEs

• <u>Hydrostatic Equilibrium in GR</u>

→ Generalize $\frac{dP}{dr} = -\rho \frac{GM}{r^2}$

(1) $\rho \,\hat{=}$ "measure of inertia" → inertial mass density

→ in relativity



P

→ total energy density

$\rho + \frac{P}{c^2}$

→ Replace $\rho \longrightarrow \rho + \frac{P}{c^2} = \rho\left(1 + \frac{P}{\rho c^2}\right)$

(2) $m \,\hat{=}$ 'source of gravity'

→ Replace $m \longrightarrow m_{eff} = \frac{4\pi r^3}{3}\rho_{eff}$

$$= \underbrace{\frac{4\pi r^3}{3}\rho}_{m} + 4\pi r^3 \frac{P}{c^2}$$

$$\Rightarrow m_{eff} = m\left(1 + \frac{4\pi r^3 P}{mc^2}\right)$$

(3) space in gravitational field curved

$$\Rightarrow \text{Replace } r^2 \rightarrow r^2\left(1 - \frac{2GM}{c^2 r}\right)$$

$$\Rightarrow \frac{dP}{dr} = -\rho\left(1 + \frac{P}{\rho c^2}\right)\frac{GM\left(1 + 4\pi r^3 \frac{P}{mc^2}\right)}{r^2\left(1 - \frac{2GM}{rc^2}\right)}$$

$$= -\rho \frac{GM}{r^2}\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1}$$

"Oppenheimer - Volkoff (OV) equation"

Note: recover Newtonian form

(1) $c \rightarrow \infty$ (Newton's theory is "action at a distance")

(2) for weak gravitational fields $\left(\frac{GM}{c^2 r} \ll 1\right)$

and pressure not too strong $\left(P \ll \rho c^2\right)$

→ OV equation describes the structure of neutron stars