General Relativity (cont'd) Feb. 14, 2008

• Spacetime

- Minkowski's BIG discovery

$$(\Delta x')^2 = \frac{(\Delta x - v\Delta t)^2}{1 - \frac{v^2}{c^2}}, \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z,$$

$$c^2(\Delta t')^2 = \frac{(\Delta t - \frac{v}{c^2}\Delta x)^2}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - c^2(\Delta t')^2 = \underbrace{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2}_{\boxed{\text{"spacetime interval"}}}$$

$\Rightarrow$ 4-dimensional spacetime

$\rightarrow$ spacetime interval is invariant - does not depend on the observer.

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + \underbrace{dy^2 + dz^2}_{\text{often assumed to be zero}}}$$

Compare with Pythagorean Thm. in 3-D space

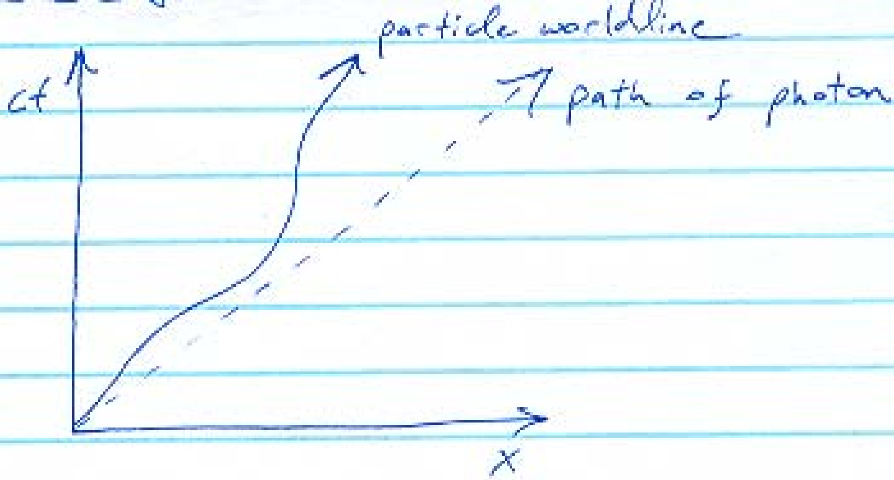
Q: What is spacetime?

- 4 dimensional (t, x, y, z)

- real arena for all of physics

- collection of all events

- spacetime diagram



- characterize spacetime interval in a neat way

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$$

$\mu, \nu \rightarrow 0, 3$

$dx^0 = c\,dt$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \hat{=} \text{"Minkowski metric"}$$

→ write even neater — $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

Einstein summation rule:
"Sum over repeated indices"

→ Spacetime in special relativity (SR) is "flat"
- Metric coefficients (here $\eta_{\mu\nu}$) do not depend on location;
$\eta_{\mu\nu} = \text{const.}$

→ In spacetime, have 4-vectors
(same for all observers)

- e.g. $x^\mu = (\underset{\substack{\uparrow \\ \text{time part}}}{ct}, \underbrace{x, y, z}_{\text{space part}}) = (ct, \vec{r})$

$V^\mu = \gamma(c, \vec{v})$ "4-velocity"

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \hat{=}$ Lorentz factor

$$p^{\mu} = \left(\frac{E}{c}, \vec{p} \right)$$ "4-momentum"

Q: What do we gain?

A: Method to write laws of physics in an invariant way

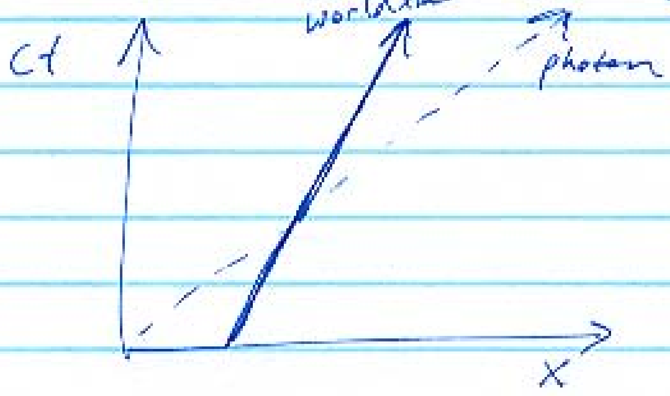
$$V^{\mu} = \frac{dx^{\mu}}{d\tau} \quad ; \quad p^{\mu} = m_0 V^{\mu}$$

• Equation of motion

Newton vs. Einstein (SR)

$$m \frac{d\vec{r}^2}{dt^2} = \vec{F} \qquad\qquad m_0 \frac{d^2 x^{\mu}}{d\tau^2} = \underbrace{f^{\mu}}_{\text{"4-force"}}$$

→ force-free motion in SR ($f^{\mu} = 0$)



$$\boxed{\frac{d^2 x^{\mu}}{d\tau^2} = 0}$$

→ Einstein's great idea: gravity is curvature of spacetime!

→ Q: How can we relate gravity to geometry of spacetime?

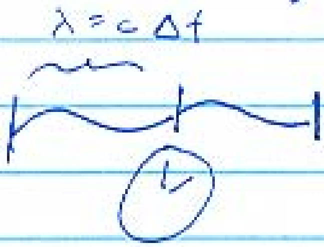
→ Begin by reformulating Newton's theory in language of spacetime!

→ recall: gravitational redshift

$$\nu_\infty = \nu_{em} \left(1 - \frac{GM}{Rc^2}\right)$$

→ tells us about the flow of time!

→ consider: 'light-clock'



$$\lambda = \frac{c}{\nu} = c\,\Delta t$$

$$\frac{1}{\Delta\tau_\infty} = \frac{1}{\Delta\tau_{em}} \left(1 - \frac{GM}{rc^2}\right)$$

$$\Longrightarrow \Delta\tau_{em} = \Delta\tau_\infty \left(1 - \frac{GM}{rc^2}\right)$$

"clock on surface of star goes slower!"