Test #1
Solutions

$\boxed{1}$ a) The free-fall timescale implies the density of a star, and also characterizes the timescale on which dynamical changes occur, such as pulsations and gravitational collapse.

b) The maximum mass that a white dwarf may have - the Chandrasekhar Mass.

c) Pressure is a consequence of energy (units are of energy density), and in GR both mass and energy are sources of gravity.

d) The pressure exerted by electrons packed tightly together enough that the Pauli Exclusion principle plays an important role in determining their bulk properties - degeneracy pressure.

e) The Hertzsprung-Russell Diagram is a plot displaying the relation of temperature to luminosity for stars.

f) Photons are redshifted as they lose kinetic energy in climbing out of a gravitational potential well - gravitational redshift.

g) i) White dwarf - R ~ 10,000 km

ii) Neutron star - R ~ 10 km

h) The Fermi Momentum is the characteristic momentum of fermions packed together maximally in phase-space, as dictated by the Pauli Exclusion Principle and the Heisenberg Uncertainty Principle.

i) Spacetime is a 4-dimensional construct unifying the three spatial dimensions and time.

j) A black hole with mass equal to the Planck Mass has a position that is not defined to be within its own Schwarzschild radius, while GR demands that nothing interior to this radius can ever be found outside of it.

$\boxed{2}$ a) $$\tau_{ff} \sim \sqrt{\frac{1}{G\rho}}$$

$$= \sqrt{\frac{1}{G n m_H}}$$

$$\sim 10^8 \text{ yr.}$$

b) Minkowski said this!

c) The proper time for a photon never advances, in any frame – $d\tau = \frac{ds}{c} = 0$ always.

Thus, the clock registers no passage of time.

$\boxed{3}$ a) Integration yields

$$\int_0^{P(r)} dP = \int_R^r -\frac{4\pi}{3} G \rho_c^2 \, r \exp\left(\frac{-r^2}{a^2}\right)$$

$$\Longrightarrow P(r) = \frac{2\pi}{3} G a^2 \rho_c^2 \exp\left(\frac{-r^2}{a^2}\right)\Bigg|_R^r$$

$$= \frac{2\pi}{3} G a^2 \rho_c^2 \left[\exp\left(\frac{-r^2}{a^2}\right) - \exp\left(\frac{-R^2}{a^2}\right)\right]$$

For $P_c = P(0)$, we find

$$P_c = \frac{2\pi}{3} G a^2 \rho_c^2 \left[1 - \exp\left(\frac{-R^2}{a^2} \right) \right],$$

which for $R = 2 R_\odot$ gives $P_c \sim 10^{18}$ Pa.

b) As $r \to 0$, the mass enclosed $M(r) \longrightarrow 0$, which implies that the gravitational force balanced by the pressure gradient goes to zero, as well. Thus, the pressure gradient must disappear, too.

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4 a) Just roughly, we have

$$E_{pot} \sim \frac{G M^2}{R}$$

$$\sim \frac{7 \times 10^{-11} \times 4 \times 10^{60}}{7 \times 10^{6}}$$

$$\sim 4 \times 10^{43} \text{ erg}$$

b) Similarly, $P_c \sim \frac{G M^2}{R^4} \sim 10^{23}$ Pa.

c) Once again very roughly, we balance the pressure from part b with the ideal gas pressure, $P = n k_B T$. We estimate that

$$n \sim \frac{M}{m_H R^3},$$ so that we obtain

$$T \simeq \frac{G M m_H}{k_B R}$$

$$\sim \frac{10^{-10}\, 10^{30}\, 10^{-27}}{10^{-23}\, 10^{7}} \sim 10^{9}\ K$$

$\boxed{5}$ a) With respect to the coordinate time, t, measured at $r \to \infty$, a clock at radius r will read

$$\tau = \left(1 - \frac{2GM}{rc^2}\right) t.$$

Therefore, the ratio of the proper times at two radii, r_{out} and r_{in}, will be

$$\frac{\tau_{out}}{\tau_{in}} = \frac{\left(1 - \frac{2GM}{r_{out}c^2}\right)}{\left(1 - \frac{2GM}{r_{in}c^2}\right)}.$$

In the case here $r_{in} = 5 R_\odot$, $r_{out} = 10^{11}$ m, and $\tau_{in} = 1$ sec. Thus, here

$$\tau_{out} = 1.0001 \text{ sec}.$$

b) Rewriting the metric, we have

$ds^2 = -c^2 d\tau^2$, which yields

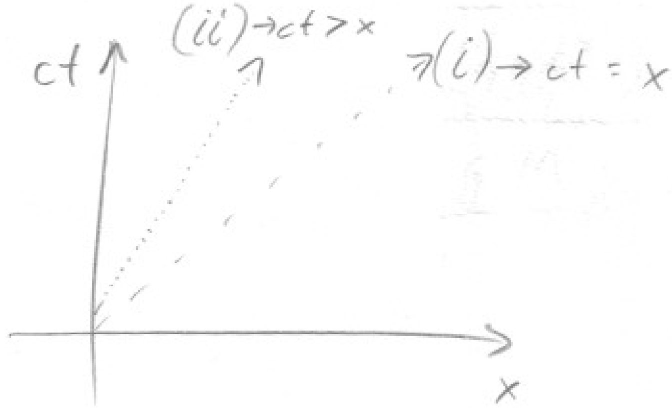
$$d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \frac{1}{c^2}\left(1 + \frac{2GM}{rc^2}\right)\left(dx^2 + dy^2 + dz^2\right).$$

If $c \to \infty$, then we obtain

$d\tau^2 = dt^2 \Rightarrow d\tau = dt$, which is interpreted to mean that there is a universal time measured to be the same by all observers, independent of r, x, y, or z.

$\boxed{6}$ a) $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

b)



c) An observer will see the photon traveling, say, in the x-direction with speed c, such that $dx = c\,dt$ (with $dy = dz = 0$). Then,

$$
\begin{aligned}
ds^2 &= -c\,dt^2 + dx^2 \\
&= -c\,dt^2 + c^2 dt^2 \\
&= 0.
\end{aligned}
$$