3/6/07

GR : Brief intro. (cont'd)

$$\ddot{x}^{\mu} + \underbrace{\frac{1}{2} g^{\mu\lambda}\left(\frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} + \frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}}\right)}_{\Gamma^{\mu}_{\alpha\beta}\ \equiv\ \text{Christoffel symbols}} \dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

$$\Rightarrow \boxed{\ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0}$$

→ Law of motion in Einstein's theory

→ "Geodesic Equation"

"Curved space tells matter how to move"

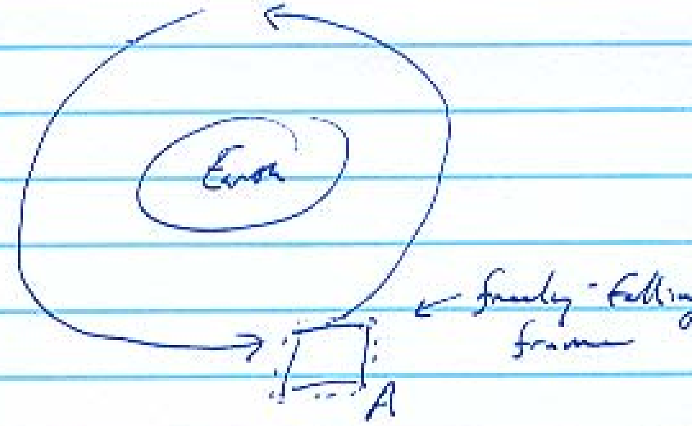
E.g. (i) No gravity → no curvature (SR)

$$\frac{\partial g_{\alpha\lambda}}{\partial x_{\beta}} = 0 \quad \Rightarrow \Gamma^{\mu}_{\alpha\beta} = 0$$

$$\Rightarrow \ddot{x}^{\mu} = 0 \quad , \quad \frac{d^2 x^{\mu}}{d\tau^2} = 0$$

(ii) in GR, consider locally freely-falling frame

$$\left.\underbrace{\frac{dx^\mu}{d\tau^2} = 0}_{SR}\right|_A$$



(iii) Gravity is weak $|\varphi| << c^2 \rightarrow v << c$
($\rightarrow$ Solar system)

$$\frac{d^2x^i}{dt^2} \simeq -\frac{\partial\varphi}{\partial x_i} \qquad i = \overset{x}{1}, \overset{y}{2}, \overset{z}{3}$$

$\rightarrow$ Recovered Newton's Law of motion

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\varphi$$

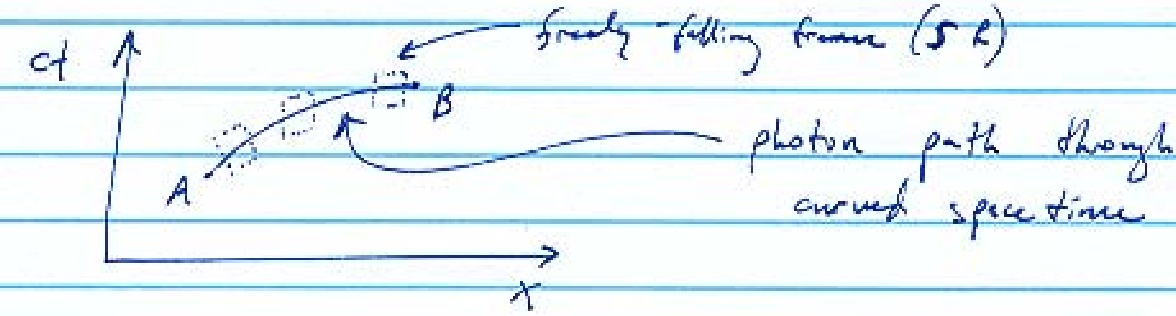
<u>Motion of photons</u>

Consider photon motion in SR
$\rightarrow$ along x-axis ($dy = dz = 0$)

$$dx = c\,dt$$

- Consider spacetime interval: $ds^2 = -c\,dt^2 + dx^2$
$= 0$

$\Longrightarrow$ Photons move along "Null-geodesics"
- Same is true in GR



SR locally $\Longrightarrow$ GR globally

$ds = 0 \qquad\qquad \sum ds = 0$

Relativistic Field Equation

$\rightarrow$ recall Newton's $\rightarrow \nabla^2 \varphi = 4\pi G \rho$

- Newtonian spacetime, $g_{00} = -\left(1 - \frac{2GM}{rc^2}\right)$

$$= -\left(1 + \frac{2\varphi}{c^2}\right)$$

$$\Longrightarrow \nabla^2 g_{00} = \frac{-2}{c^2} \nabla^2 \varphi$$

$$= \frac{-8\pi G}{c^2} \rho$$

- Q: What are sources of gravity?

  - Newton: ρ (mass density)
  - Einstein: ρc^2 (mass energy density)

    pressure, P, has units of energy density

→ Neatly organize all sources of gravity

$$T_{\mu\nu} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \rho c^2 & & & \\ 1 & & P & & \\ 2 & & & P & \\ 3 & & & & P \end{array}$$

("stress-energy tensor")

→ Define 'effective density' → total strength of gravity

$$c^2 \rho_{eff} = \left(T_{00} + T_{11} + T_{22} + T_{33} \right) = \rho c^2 + 3P$$

$$\Longrightarrow \boxed{\rho_{eff} = \rho + \frac{3P}{c^2}}$$

With $T_{00} = \rho c^2 \Rightarrow \nabla^2 g_{00} = \frac{-8\pi G}{c^2} T_{00}$

- In general,

$$\nabla^2 g_{00} \xrightarrow{\text{replace}} \underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} = f\left(g_{\mu\nu},\ \text{1st derivative of } g_{\mu\nu},\ \text{2nd derivative of } g_{\mu\nu}\right)$$

$$\Longrightarrow \boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \quad (\text{"Field equation"})$$

schematically:

$$\begin{pmatrix}\text{measure of} \\ \text{curvature of} \\ \text{spacetime}\end{pmatrix} = \frac{8\pi G}{c^4} \begin{pmatrix}\text{measure of} \\ \text{energy-density}\end{pmatrix}$$

"Matter tells space how to curve"