GR: Brief Intro. (cont'd)

Q: How can we relate gravity to the curvature of spacetime?

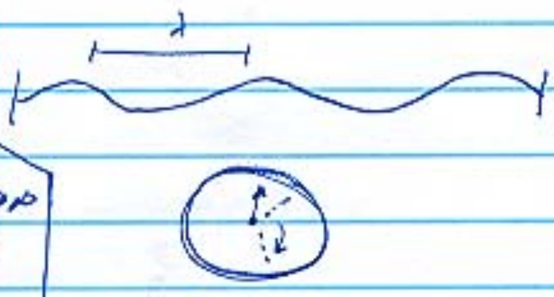
- Reformulate old Newtonian gravity in a new language

- Recall: gravitational redshift ($\rightarrow$ Newtonian effect)

$$\nu_\infty = \nu_{em}\left(1 - \frac{GM}{Rc^2}\right)$$

$\longrightarrow$ Tells us also about the flow of time

- Consider a 'light clock'



$$\lambda = c\,\Delta t$$

$$\Rightarrow \nu = \frac{c}{\lambda}$$

$$\Rightarrow \nu \propto \frac{1}{\Delta t}$$

$$\Longrightarrow \frac{1}{\Delta t_\infty} = \frac{1}{\Delta t_{em}}\left(1 - \frac{GM}{rc^2}\right)$$

$$\Rightarrow \Delta t_{em} = \Delta t_\infty\left(1 - \frac{GM}{rc^2}\right)$$

Recall: proper time $\equiv$ time measured by clocks at rest

Δt_{em} = proper time measured at radius r

i.e., $\Delta t_{em} = \Delta\tau$

$\Delta t_\infty = \Delta t$ (time measured at $r \to \infty$)

↑ 'coordinate time'

$$\Longrightarrow \quad \Delta\tau = \left(1 - \frac{GM}{Rc^2}\right)\Delta t$$

- connect to spacetime interval

$$ds^2 = -c^2 d\tau^2 \qquad \left(\begin{array}{c} dx = dy = dz = 0, \\ \text{in clock rest frame} \\ \text{at } r \end{array}\right)$$
$$= -c^2\left(1 - \frac{GM}{rc^2}\right)^2 dt^2$$

- Consider weak gravitational fields $\left(\frac{GM}{rc^2} \ll 1\right)$

Then,

$$\left(1 - \frac{GM}{rc^2}\right)^2 = 1 - \frac{2GM}{rc^2}$$

⟹ Spacetime interval for Newtonian gravity

$$ds^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{\text{Pythagoras}}$$

Recall: Newtonian gravitational potential

$$\varphi = \frac{-GM}{r}$$

metric coefficient $\longrightarrow$ φ(gravity)

$$\left(1 + \frac{2\varphi}{c^2}\right)$$

In general: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$g_{\mu\nu}$ ≡ "metric tensor"
↪ a special matrix

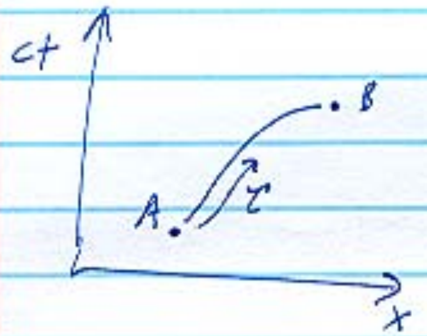
e.g. weak field metric

$$g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right); \quad g_{11} = g_{22} = g_{33} = 1;$$

all others zero.

Motion of particles through (curved) spacetime

- <u>Rule</u>: particles move through spacetime along straightest possible lines ($\rightarrow$ geodesics)



$\longrightarrow$ find world line $x^\mu = \begin{pmatrix} ct(\tau) \\ x(\tau) \\ y(\tau) \\ z(\tau) \end{pmatrix}$

$\boxed{\text{all functions of } \tau}$

$$\int_A^B ds = \text{extremal} \quad \underset{\text{calculus of variations}}{\Longrightarrow} \quad \delta \int_A^B ds = 0$$

Rephrase as $\quad ds = \left(g_{\mu\nu}\, dx^\mu dx^\nu\right)^{1/2}$

$$= \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\right)^{1/2} d\tau$$

$$L(x^\mu, \dot{x}^\mu) = \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{1/2}$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

$$\Longrightarrow \int_A^B L(x^\mu, \dot{x}^\mu)\, d\tau = \text{extremal}$$

solve Euler-Lagrange equation:

$$\boxed{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}}$$

Evaluate this equation:

$$\boxed{\ddot{x}^\mu + \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\lambda}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \right) \dot{x}^\alpha \dot{x}^\beta = 0}$$