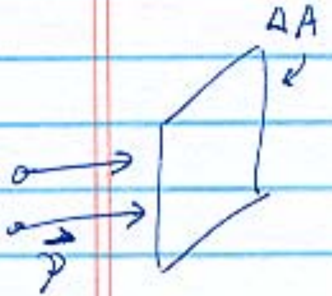
1/30/07

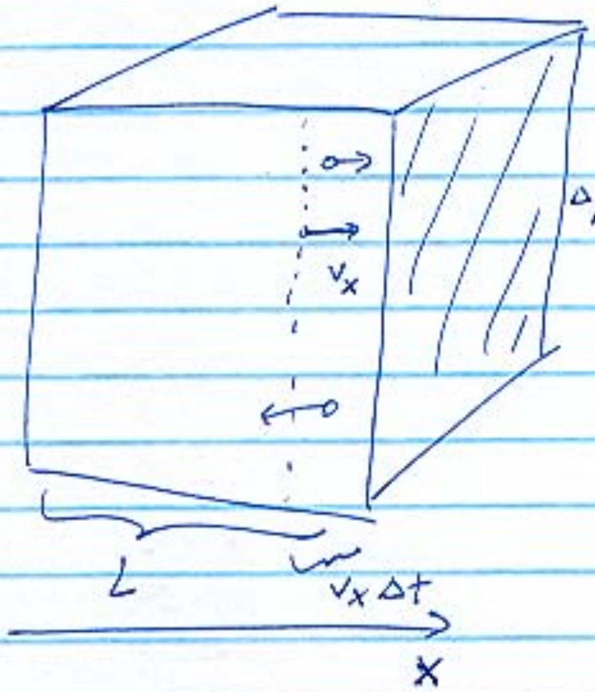
Physics of compact objects (con't)

Pressure, P:

$$P = \frac{\Delta p}{\Delta A \, \Delta t} \quad (\text{"momentum flux"})$$



Consider a box, with side length L, filled with particles:



$\Delta A = L^2$

$$V = L^3$$

$$N = n L^3$$

N: total # of particles; n: number density of particles

$$\Delta V = \Delta A \, v_x \, \Delta t$$

$$\Delta N = \frac{1}{2} n \, \Delta V$$

$$= \frac{1}{2} n \, \Delta A \, v_x \, \Delta t$$

only ½ of the particles in the volume ΔV travel in the $+x$ direction

Momentum in x-direction:

$$p_x = m v_x ,$$

where m is the mass of a particle.

$$\Delta p_x = 2\,\Delta N p_x = n\,\Delta A\, p_x\, v_x\, \Delta t$$

multiply by 2 because the particles bounce off the 'wall of the box' and their velocity goes toward the negative x-direction, i.e. $\Delta p_x = 2 p_x$ for each particle

Pressure, P, is thus:

$$P = \frac{\Delta p_x}{\Delta A\, \Delta t} = n \langle p_x v_x \rangle$$

Now, because the pressure is isotropic

$$\langle \vec{p} \cdot \vec{v} \rangle = \langle p_x v_x \rangle + \langle p_y v_y \rangle + \langle p_z v_z \rangle$$

$$= 3 \langle p_x v_x \rangle$$

$$\Longrightarrow P = \frac{n}{3} \langle \vec{p} \cdot \vec{v} \rangle$$

Pressure in an ideal gas:

- non-relativistic (NR) particles, i.e.

$$\epsilon_{kin} \ll m_0 c^2$$

$$\left.\begin{array}{l} p = m_o v \\ \epsilon_{kin} = \frac{1}{2} m_o v^2 \end{array}\right\} \Rightarrow pv = 2\,\epsilon_{kin}$$

$$\Longrightarrow \quad P = \frac{1}{3}\frac{N}{V}\langle \vec{p}\cdot\vec{v}\rangle$$

$$= \frac{2}{3}\frac{N}{V}\epsilon_{kin} = \frac{2}{3}U_{kin}, \quad \text{where}$$

$U_{kin} \equiv$ kinetic energy density

<u>Now, for ultra-relativistic particles (UR):</u>

$$\epsilon_{kin} \gg m_o c^2 \qquad v \simeq c$$

$$\Rightarrow \quad \epsilon_{kin} \simeq \underbrace{\left(m_o^2 c^4 + p^2 c^2\right)^{1/2}}_{\epsilon_{tot}} \simeq pc$$

$$\Longrightarrow \quad pv = \epsilon_{kin}, \quad \text{and}$$

$$P = \frac{1}{3}\frac{N}{V}\langle \vec{p}\cdot\vec{v}\rangle = \frac{1}{3}\frac{N}{V}\epsilon_{kin} = \frac{1}{3}U_{kin}$$

<u>Total Energy and Stability:</u>

Consider NR gas:

For stability: $\langle P \rangle = \frac{-1}{3} \frac{E_{pot}}{V}$

We have that $\langle P \rangle = \frac{2}{3} \langle U_{kin} \rangle$

$$= \frac{2}{3} \frac{E_{kin}}{V},$$

where $E_{kin} \hat{=}$ total kinetic energy.

In hydrostatic equilibrium:

$$\frac{2}{3} \frac{E_{kin}}{V} = \frac{-1}{3} \frac{E_{pot}}{V}$$

$$\Longrightarrow \boxed{2 E_{kin} = - E_{pot}},$$

often called the virial theorem. (VT)

Total energy, $E_{tot} = E_{kin} + E_{pot}$

If system is gravitationally bound, then

$E_{tot} < 0$, since one must have no velocity outwards, v, as $R \rightarrow \infty$:

$$E_{tot} = \frac{1}{2} m v^2 - \frac{G m^2}{R}.$$

Now, $\boxed{E_{tot} = -2 E_{kin} + E_{kin} = - E_{kin}}$, from the virial theorem.

$\Rightarrow$ NR gas can be stable.

<u>Consider UR gas:</u>

For stability: $\langle P \rangle = \frac{-1}{3} \frac{E_{pot}}{V}$

We have that $\langle P \rangle = \frac{1}{3} \frac{E_{kin}}{V}$

$$\Longrightarrow \boxed{E_{tot} = E_{pot} + E_{kin} = 0}$$