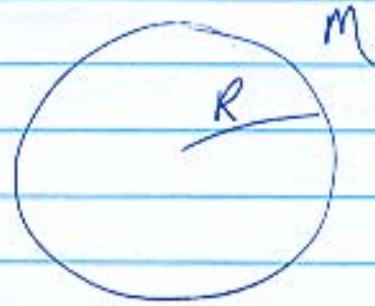
1/25/07

Physics of Compact Objects

Theorists' star:

1) Perfect sphere
2) No rotation
3) No $\vec{B}$ - field



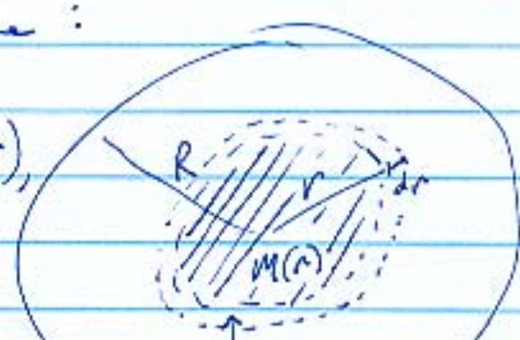
Solar values $\rightarrow$ $\left[\begin{array}{l} M_\odot = 2 \times 10^{30} \text{ kg} \\ R_\odot = 6.96 \times 10^{8} \text{ m} \end{array} \right.$

Average density: $$\langle \rho \rangle_\odot = \frac{M_\odot}{\frac{4\pi}{3} R_\odot^3}$$

$$= 1,400 \text{ kg m}^{-3}$$

Mechanical structure:

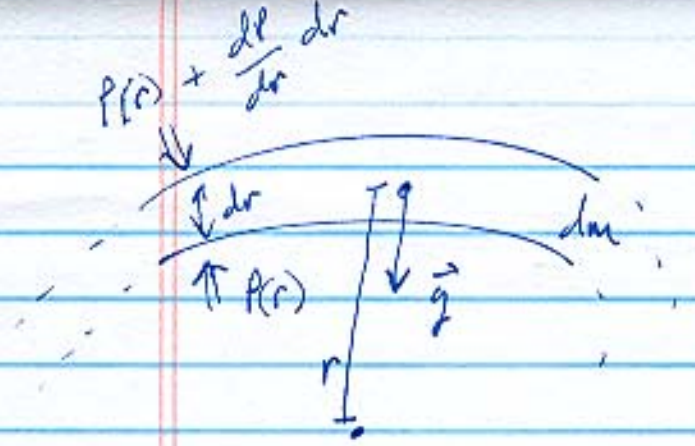
Mass coordinate: $m(r)$, mass enclosed at radius r. Mass in a shell of thickness dr:



$$dm = dV \rho(r) = 4\pi r^2 dr \, \rho(r)$$

$$M(r) = \int_0^r 4\pi r'^2 \rho \, dr'$$

Equation of motion: total force = mass × acceleration



2 Forces:

$$\text{gravity} = -g\,dm$$

$$= \frac{-G\,m(r)\,dm}{r^2}$$

$$\text{pressure} = 4\pi r^2 P(r) - 4\pi r^2 \left[P(r) + \frac{dP}{dr} dr\right]$$

Total force = gravity + pressure

$$= -g\,dm - \frac{1}{\rho}\frac{dP}{dr}\underbrace{4\pi r^2 \rho\, dr}_{dm}$$

$$= \left[-g - \frac{1}{\rho}\frac{dP}{dr}\right] dm$$

Equation of motion $\Rightarrow$ $\boxed{\frac{d^2 r}{dt^2} = -g - \frac{1}{\rho}\frac{dP}{dr}}$

A thought experiment:
Case w/ pressure = 0.

$$\frac{d^2 r}{dt^2} = -g \Rightarrow -\frac{R}{\tau_{ff}^2} = -\frac{GM}{R^2}$$

$$\Rightarrow \tau_{ff} = \frac{1}{\sqrt{G\frac{M}{R^3}}} \simeq \frac{1}{\sqrt{G\rho}}, \text{ free-fall time}$$

For the sun, $\tau_{ff} \sim 1$ hour!

Thus, pressure must be countering gravity in stable stars:

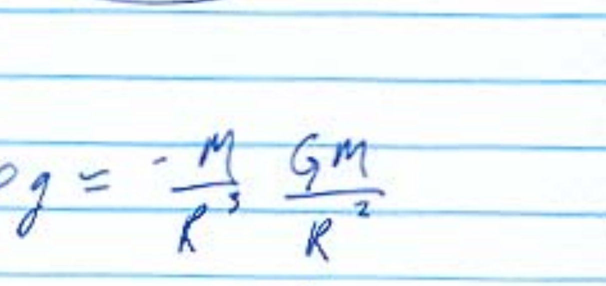
$\boxed{\text{pressure} \simeq \text{gravity}}$.

$\Rightarrow$ For stable stars, $\frac{d^2r}{dt^2} = 0$, so that

$$\frac{dP}{dr} = -\rho g$$

A simple estimate for a star's central pressure, P_c.

$$\frac{dP}{dr} = \frac{P_c - \cancelto{0}{P_0}}{0 - R}$$



$$\simeq -\frac{P_c}{R}$$

$$\Rightarrow -\frac{P_c}{R} \simeq -\rho g. \text{ Now, } -\rho g \simeq -\frac{M}{R^3}\frac{GM}{R^2}$$

$$= -\frac{GM^2}{R^5}.$$

Thus, $P_c \simeq \frac{GM^2}{R^4}$, which is a good approximation, especially for the case of the Sun.

<u>A different look at hydrostatic equilibrium:</u>

Total gravitational potential energy

$$E_{pot} = -\int_0^R \frac{G M(r)}{r}\, dm$$

Find the average pressure needed to balance gravity:

- Calculate average pressure

$$\langle P \rangle = \frac{\int_0^R 4\pi r^2 P(r)\, dr}{\int_0^R 4\pi r^2\, dr} = \frac{\int_0^R 4\pi r^2 P(r)\, dr}{\underset{\nwarrow \text{volume of star}}{V}}$$

Now, integrate by parts to obtain

$$\int_0^R 4\pi r^2 P(r)\, dr = \underbrace{\cancel{\frac{4\pi}{3}\left[P r^3\right]_0^R}}_{\text{At the surface, pressure is negligible.}}{}^{\to 0} - \frac{1}{3}\int_0^R 4\pi r^3 \frac{dP}{dr}\, dr$$

Use hydrostatic equilibrium:

$$4\pi r^3 \frac{dP}{dr} dr = -4\pi r^3 \rho \frac{G\,m(r)}{r^2} dr$$

$$= -4\pi r^2 \rho\, dr \; \frac{G\,m(r)}{r}$$

$$= -\frac{G\,m(r)}{r} dm$$

Use this in previous equation to get

$$\int_0^R 4\pi r^2 P\, dr = \frac{1}{3} \int_0^R \frac{GM}{r} dm$$

$$= -\frac{1}{3} E_{pot}$$

$$\Rightarrow \boxed{\langle P \rangle = -\frac{1}{3} \frac{E_{pot}}{V}}$$ is the average pressure.