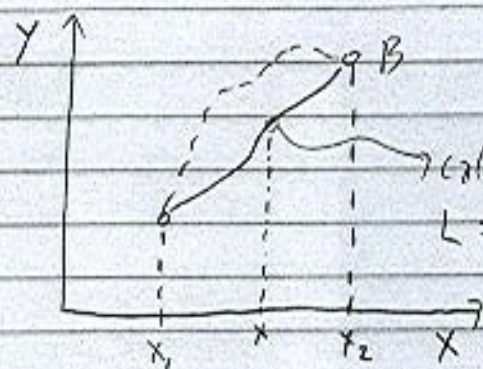


Calculus of Variations: Brief Introduction

①

(1) Basic Method:

- Consider paths in x - y plane from point A to B :



calculate value of some function $L = L(y, y')$ for each x along the path!

$$\left(y' = \frac{dy}{dx} \right)$$

- Take integral along path

$$I = \int_{x_1}^{x_2} L(y(x), y'(x)) dx$$

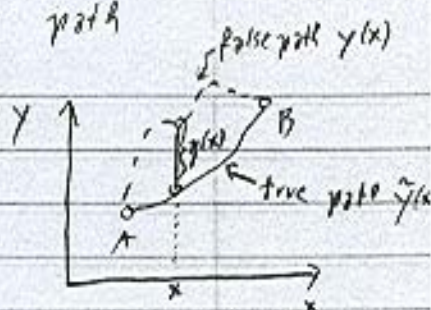
- For different paths $y(x)$, one gets different values of I

- Ask: For what path $\tilde{y}(x)$ is integral minimum (or maximum) in general; an extremum?

- Consider difference between (unknown) true path

and a nearby path:

$$y(x) = \tilde{y}(x) + \eta(x)$$



$$\Rightarrow y'(x) = \tilde{y}'(x) + \eta'(x), \text{ where } \eta'(x) = \frac{d\eta}{dx}$$

→ the $\eta(x)$ is an arbitrary function, which is very small compared to $\tilde{y}(x)$ [$|\eta(x)| \ll |\tilde{y}(x)|$]

- now evaluate the difference:

$$\Delta I = \int_{x_1}^{x_2} [L(y, y') - L(\tilde{y}, \tilde{y}')] dx$$

[Note: in the calculus of variations, ΔI is often denoted δI]

- Taylor - expand:

$$L(y, y') = L(\tilde{y}, \tilde{y}') + \frac{\partial L}{\partial y} \eta(x) + \frac{\partial L}{\partial y'} \eta'(x) + (\text{higher-order terms in } \eta)$$

- Insert:

$$\Delta I = \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial y} \eta(x) + \frac{\partial L}{\partial y'} \eta'(x) \right] dx$$

- Use integration by parts to

rewrite $\int \frac{\partial L}{\partial y'} \eta'(x) dx$:

-> Since $\eta'(x) = \frac{d}{dx} \eta(x)$, one finds:

$$\int_{x_1}^{x_2} \frac{\partial L}{\partial y'} \eta'(x) dx = \left[\frac{\partial L}{\partial y'} \eta(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \eta(x) dx$$

- Now comes the crucial trick:

-> All paths (true or false) must go through points A and B

$$\Rightarrow \eta(x_1) = \eta(x_2) = 0$$

\Rightarrow The first term on RHS is ZERO!

- We can now write:

$$\Delta I = \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \right] \eta(x) dx$$

- To have an extremum, need: $\Delta I = 0$

(recall ordinary calculus!)

- But: Integral above is zero for arbitrary $\eta(x)$ ONLY IF

$\frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = \frac{\partial L}{\partial y}$

(This is famous Euler-Lagrange equation)

- One can make this more general:

→ assume that function L depends on more than one variable:

$$L = L(y_1, y_1', y_2, y_2', y_3, y_3', \dots) = L(y_i, y_i')$$

→ One then has an equation for each pair (y_i, y_i') :

$$\boxed{\frac{d}{dx} \left(\frac{\partial L}{\partial y_i'} \right) = \frac{\partial L}{\partial y_i}} \quad \text{for all } i$$

- Note: Be very careful when you take the partial derivatives, in terms of what to hold constant. E.g., when you take $\frac{\partial L}{\partial y_i}$, make sure to treat y_i' as a constant and vice versa!

Be also very careful when you apply the $\frac{d}{dx}$ derivative. Both y_i AND y_i' depend on x !

(2) An Example:

→ The most famous of them all:

THE BRACHISTOCHRONE* PROBLEM

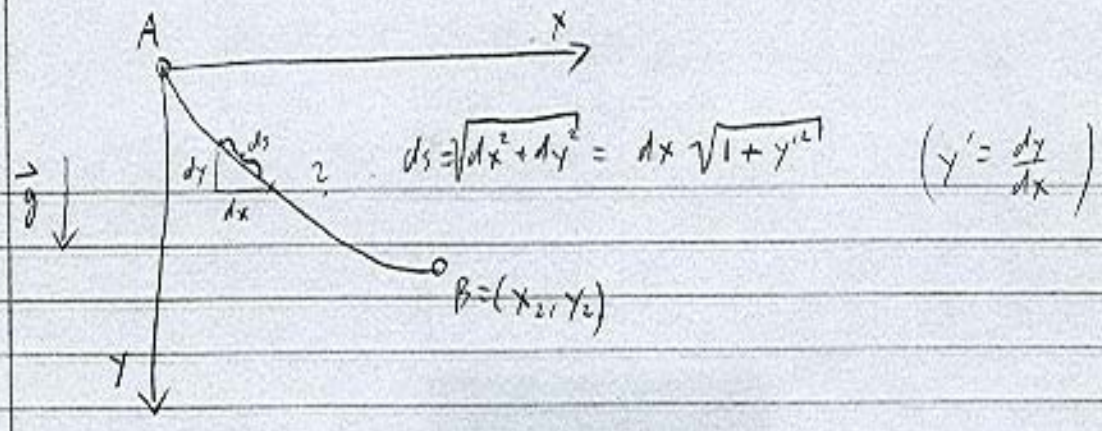
(*Greek for "shortest time")

— posed in 1696 by Johann Bernoulli:

"Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument."

— Problem: Consider a particle with mass m , which starts from rest at point A and falls under the influence of Earth's gravity, assuming that the particle is restricted to fall along a frictionless curve $y(x)$ from the origin at $A = (0, 0)$ to a final point $B = (x_2, y_2)$, which is NOT located vertically under the starting point.

Which curve, $y(x)$, yields the shortest time of descent between the fixed points A and B ?



- The time of descent is

$$t_{AB} = \int \frac{ds}{v}$$

$ds \hat{=}$ small piece along path

- How to figure out speed v ?

A: From conservation of energy ($E_{pot} + E_{kin} = \text{const.}$)

$$mgy = \frac{1}{2} m v^2 \Rightarrow v = \sqrt{2gy} \quad (g = 9.8 \text{ m/s}^2)$$

$$\Rightarrow t_{AB} = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx$$

- Define $L = L(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$

- Evaluate Euler-Lagrange equation (this takes a bit of work and patience, but is otherwise straightforward)

→ Find: $2yy'' + y'^2 + 1 = 0$

Q: How to solve this one ?

- Trick: Use the following parametric form

$$\left. \begin{aligned} x &= a(d - \sin d) \\ y &= a(1 - \cos d) \end{aligned} \right\} \text{cycloid equations}$$

The parameter d is restricted to

$$0 \leq d \leq d_0 \leq 2\pi$$

- You have to fix a and d_0 so that curve

$y(x)$ "hits" point $B = (x_2, y_2)$

-> These are the equations of (part of) a cycloid!

-> By plugging into equation, one shows that

cycloid solves $2yy'' + y'^2 + 1 = 0$, and therefore

is the desired brachistochrone solution!