2/6/07

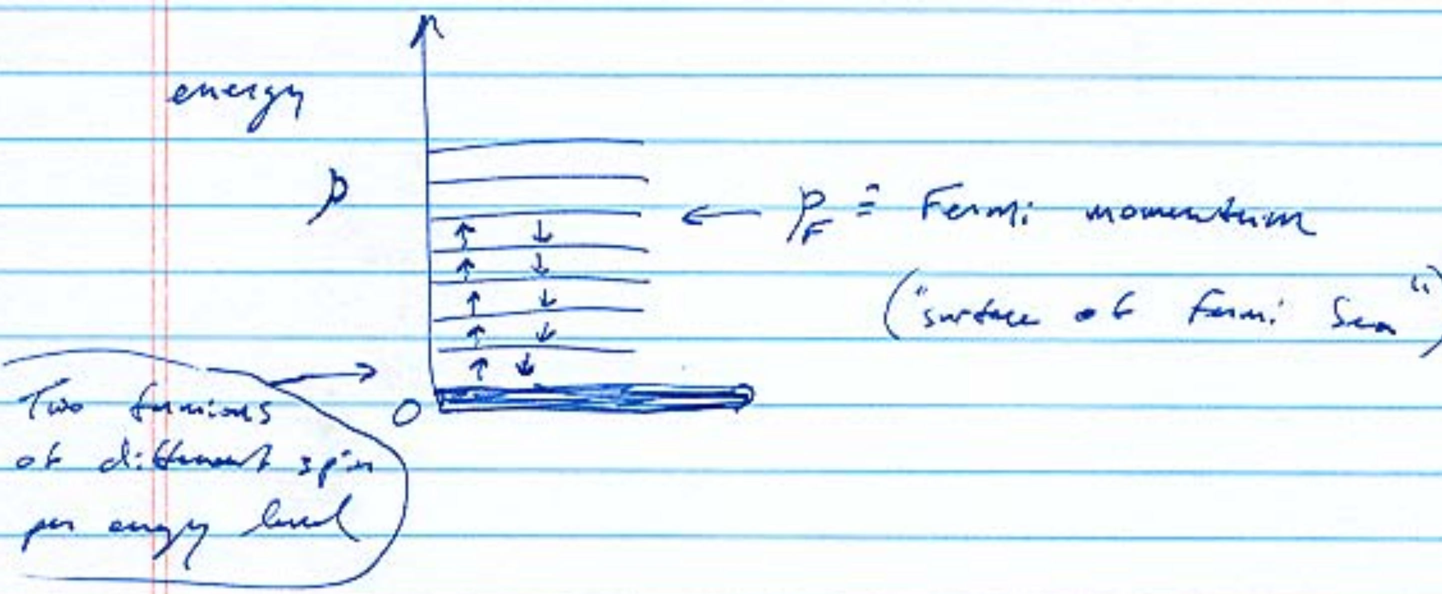
Basic Physics of Compact Objects (cont'd)

Quantum gases ("Degenerate gases")

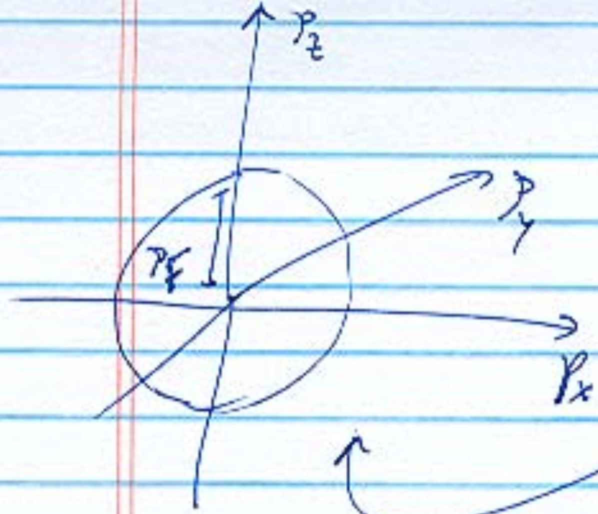
→ Pauli Exclusion Principle:
Only 2 fermions (or less) allowed per quantum cell

→ Applies to electrons, as these are fermions

Consider the case of tight packing, or "complete degeneracy":



Visualize the momentum part of the 6-dimensional phase-space



Particles will fill a sphere, with radius p_F, since there are equal probabilities of having momentum in any direction. (No preferred direction; isotropic distribution)

Q: How many particles can we fit into maximally dense phase-space?

A:

$$N = \underbrace{2}_{\text{Pauli Principle}} \frac{\frac{4\pi}{3} p_F^3 V}{h^3} = \frac{8\pi}{3h^3} p_F^3 V$$

$\frac{V}{h^3} \rightarrow$ Number density in maximally ~~[illegible]~~ degenerate particles in phase-space (From Heisenberg's Uncertainty)

$$\Rightarrow \boxed{p_F = h\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3}}, \quad \text{where } n = \frac{N}{V},$$

is the Fermi momentum.

Case 2: Quantum + NR

- gas in NR $\rightarrow$ $P = \frac{2}{3} U_{kin}$

  U_{kin} - kinetic energy density

- Estimate U_{kin}

$$\rightarrow U_{kin} \simeq n \langle \epsilon_{kin} \rangle$$

$$\langle \epsilon_{kin} \rangle \simeq \frac{p_F^2}{2m_0} \simeq \frac{p_F^2}{m_0}$$

$$\Rightarrow U_{kin} \simeq \frac{1}{h^3 m_0} p_F^5, \quad \text{as} \quad n \simeq \frac{1}{h^3} p_F^3$$

- Exact solution is $U_{kin} = \frac{8\pi}{10} \frac{p_F^5}{h^3 m_0}$

$$\Rightarrow P = \frac{2}{3} U_{kin} = \frac{h^2}{5m_0}\left(\frac{3}{8\pi}\right)^{2/3} n^{5/3}$$

$$\Rightarrow \boxed{P = K_{NR}\, n^{5/3}}$$

Case 3: Quantum + UR

Gas is UR : $P = \frac{1}{3} U_{kin}$

$U_{kin} = n \langle \epsilon_{kin} \rangle$

$$\left.\begin{aligned} \langle \epsilon_{kin} \rangle &= p_F c \\ n &\simeq \frac{p_F^3}{h^3} \end{aligned}\right\} \Rightarrow U_{kin} \simeq \frac{c}{h^3} p_F^4$$

Exact result: $U_{kin} = \frac{2\pi c}{h^3} p_F^4$

$$\Longrightarrow P = \frac{1}{3} U_{kin} = \frac{hc}{4} \left(\frac{3}{8\pi}\right)^{1/3} n^{4/3}$$

$$\Longrightarrow \boxed{P = K_{UR}\, n^{4/3}}$$

Quantum NR/UR Boundary:

$$\left.\begin{aligned} v &\simeq c \\ p_F &= m_0 v \simeq m_0 c \\ p_F &\simeq h n^{1/3} \end{aligned}\right\} \Rightarrow n_{UR} \simeq \left(\frac{c m_0}{h}\right)^3$$

critical density

For electrons, $m_0 = m_e = 9.1 \times 10^{-31}$ kg

$$\Rightarrow n_{UR} \simeq 10^{35} \text{ m}^{-3}$$

$$\boxed{\rho_{UR} \simeq m_H n_{UR} \sim 10^{8} \text{ m}^{-3}}$$

$\longrightarrow$ $\rho_{sun} \simeq 10^{5}$ kg m^{-3}, for comparison.