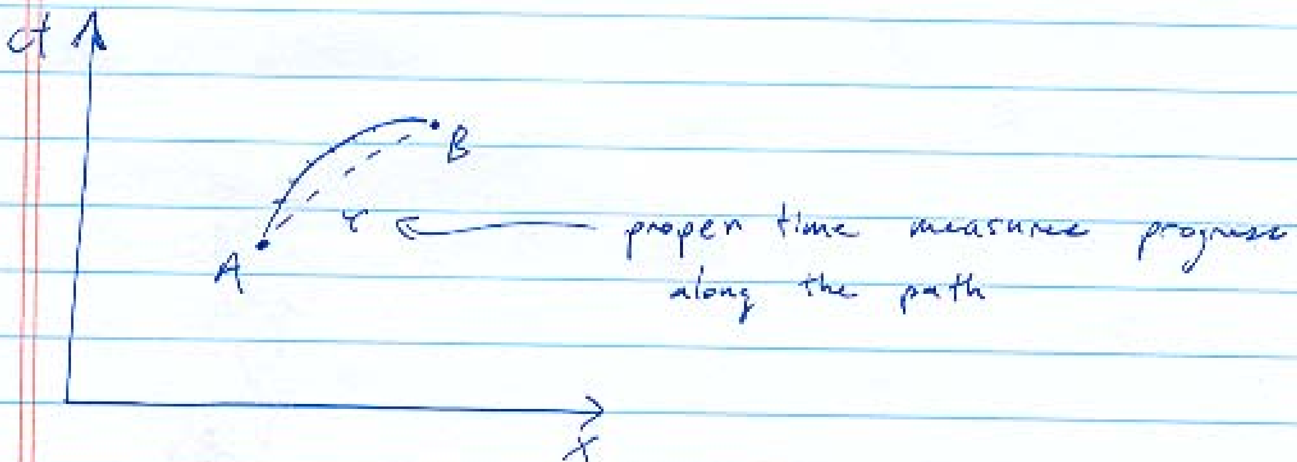
4/5/07

Black Holes (cont'd)

Geodesics: revisited

<u>Law of motion</u>: When no non-gravitational forces (e.g. electromagnetic ones) present, then all particles move on "straightest" possible paths through space time ($\rightarrow$ geodesics)

More precisely, $\int_A^B ds$ = extremum



Alternatively, $c\int_A^B d\tau$ = extremum

("principle of extremal time")

$$\Rightarrow \text{trivially,} \quad c\int_A^B \left(\frac{d\tau}{d\tau}\right) d\tau = \text{extremum}$$

Consider Schwarzschild Metric in equatorial plane ($\theta = \pi/2$, $d\theta = 0$):

$$c^2 d\tau^2 = \left(1 - \frac{R_s}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{R_s}{r}\right)} - r^2 d\varphi^2$$

$$\Rightarrow \boxed{c^2 = \left(1 - \frac{R_s}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{R_s}{r}} - r^2 \dot{\varphi}^2},$$

where $\dot{t} = \frac{dt}{d\tau}$, $\dot{r} = \frac{dr}{d\tau}$, $\dot{\varphi} = \frac{d\varphi}{d\tau}$.

Define

$$L = c\frac{d\tau}{d\tau} = L(t, \dot{t}, r, \dot{r}, \varphi, \dot{\varphi})$$

$$= \left[c^2\left(1 - \frac{R_s}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{R_s}{r}} - r^2\dot{\varphi}^2\right]^{1/2}$$

Notice: $L = c$, always!

Rephrase geodesic condition:

$$\Rightarrow \int_A^B L \, d\tau = \text{extremum}$$

Solve with calculus of variations:

$$t: \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t}$$

$$r: \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\varphi: \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}$$

Euler-Lagrange equations

Notice that for the Schwarzschild Metric,

$$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial \varphi} = 0 \qquad (t \text{ and } \varphi \text{ are 'ignorable' variables})$$

$\Longrightarrow$ Derive 2 "constants of motion"

<u>Conservation Laws in Schwarzschild Geometry</u>

① $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0$

$$\frac{\partial L}{\partial \dot{t}} = \frac{c^2\left(1-\frac{R_s}{r}\right)\dot{t}}{L}$$

$$\Rightarrow \frac{d}{d\tau}\left[c^2\left(1-\frac{R_s}{r}\right)\dot{t}\right] = 0 \quad \left(\text{recall: } L = c = \text{constant}\right)$$

$$\Rightarrow c^2\left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau} = \text{constant}$$

How to interpret this?

→ Recall: particle energy in special relativity (SR)

$$\epsilon = \gamma m_0 c^2 \quad , \text{ where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \doteq \text{"Lorentz factor"}$$

Rephrase: $-c^2 d\tau^2 = ds^2 \overset{SR}{=} -c^2 dt^2 + dx^2 = -c^2 dt^2\left(1-\frac{v^2}{c^2}\right)$

$dx = v\,dt$

$$\Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\Rightarrow \epsilon = m_0 c^2 \frac{dt}{d\tau}$$

Thus, $e = \frac{\epsilon}{m_0} = c^2\left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau} = \text{constant}$

("energy per unit rest mass")

② $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 0$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{-r^2 \dot{\varphi}}{L}$$

$\Rightarrow \frac{d}{d\tau}\left(r^2 \dot{\varphi}\right) = 0 \Rightarrow r^2 \dot{\varphi} = \text{constant}$

How to interpret?

Recall: angular momentum in Newtonian mechanics:

$J = m_0 r v \quad ; \quad v = \omega r = \dot{\varphi} r$

$= m_0 r^2 \dot{\varphi}$

$= \text{constant}$

Thus, $j = \frac{J}{m_0} = r^2 \dot{\varphi} = \text{constant}$

("angular momentum per unit rest mass")

$\Rightarrow$ Motion in Schwarzschild Geometry is such that

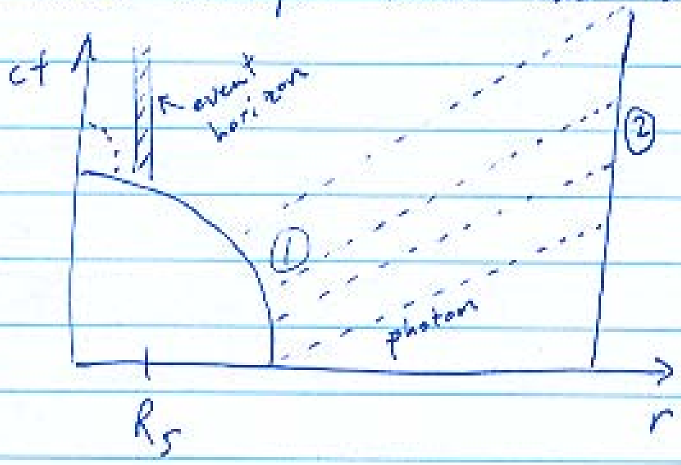
$e = \text{constant}$ (energy conservation)

$j = \text{constant}$ (angular momentum conservation)

Stellar Collapse

- Oppenheimer & Snyder (1939): $M > M_{ov}$

→ Consider collapse from two different viewpoints:



① Riding on the surface of the star

② Far-away $(r \to \infty)$