4/10/07

Black Holes (cont'd)

- Collapse of a star

→ Schwarzschild metric is applicable to EXTERIOR of even a collapsing star, although the situation is time-dependent – "Birkhoff's Theorem"

– Story told by observer ①

→ "Radial plunge" $\dot{\theta} = \dot{\varphi} = 0$

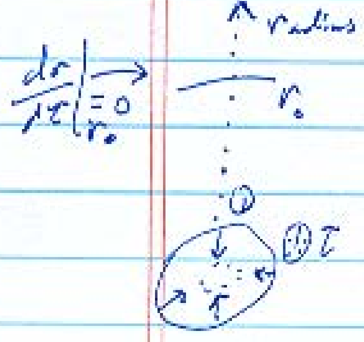
$$\Rightarrow \quad c^2 = c^2\left(1 - \frac{R_s}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{\left(1 - \frac{R_s}{r}\right)},$$

where $\dot{t} = \frac{dt}{d\tau}$ and $\dot{r} = \frac{dr}{d\tau}$.

Using also the following equation:

$$e = c^2\left(1 - \frac{R_s}{r}\right)\dot{t}$$

$$\Rightarrow \quad \frac{dr}{d\tau} = -c\left[\frac{e^2}{c^4} - 1 + \frac{R_s}{r}\right]^{1/2}$$



– $r(\tau)$ marks surface of collapsing star
– τ is proper time measured at surface

- Fix constant (e), such that plunge begins at r_0 and with zero velocity:

$$\Rightarrow \boxed{\frac{e^2}{c^4} - 1 = \frac{R_s}{r_0}}$$

$$\Rightarrow \frac{dr}{d\tau} = -c\left[\frac{R_s}{r} - \frac{R_s}{r_0}\right]^{1/2}$$

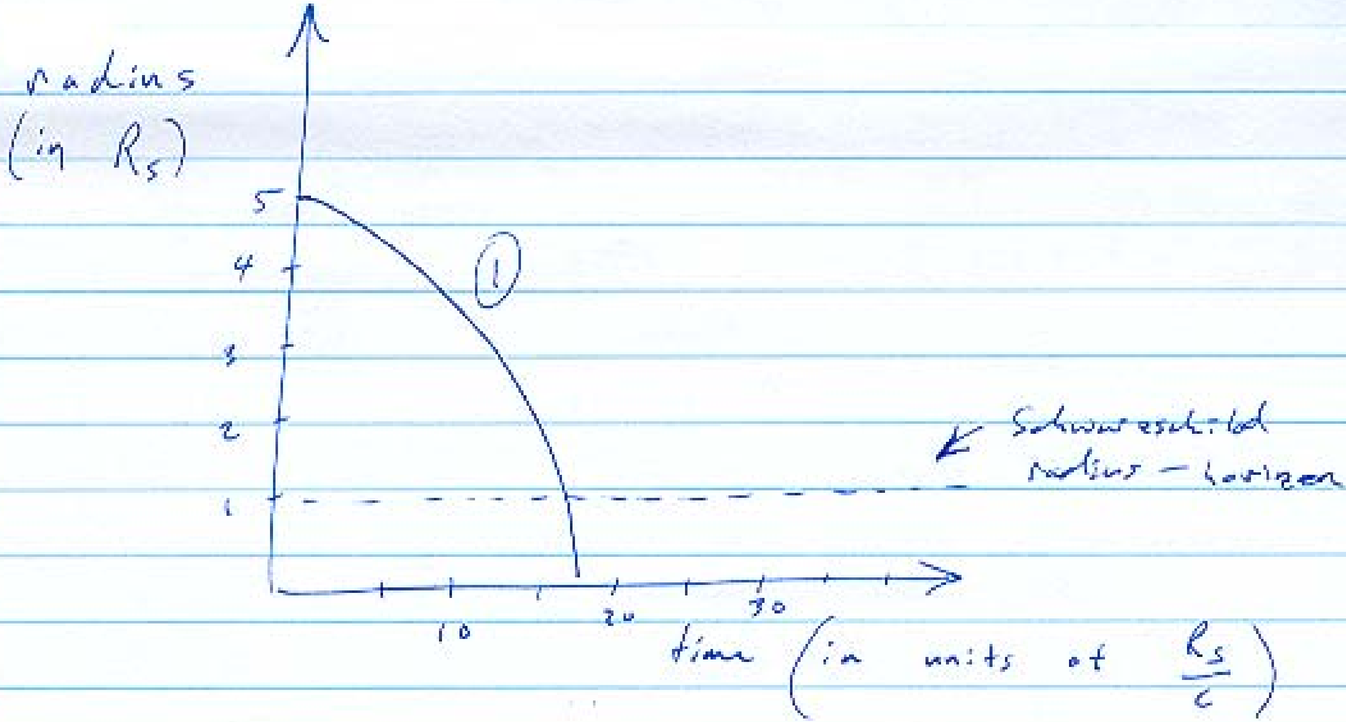
Solve by separation of variables:

$$\sqrt{\frac{r_0}{R_s}}\;\frac{dr}{\sqrt{\frac{r_0}{r} - 1}} = -c\,d\tau$$

Introduce: $\eta = \arccos\left(2\frac{r}{r_0} - 1\right)$ ("cycloid parameter")

$$\frac{d\eta}{dr} = \frac{-1}{\sqrt{1-\left(2\frac{r}{r_0}-1\right)^2}}\left(\frac{2}{r_0}\right) = \frac{-1}{r}\,\frac{1}{\sqrt{\frac{r_0}{r} - 1}}$$

Find: $\boxed{\tau(r) = \frac{1}{2}\frac{R_s}{c}\left(\frac{r_0}{R_s}\right)^{3/2}\left(\sin\eta + \eta\right)}$



$$\tau_0 \equiv \tau(r=0) = \frac{\pi}{2}\,\frac{R_s}{c}\left(\frac{r_0}{R_s}\right)^{3/2}$$

(e.g. $\tau_0 = 17.6\,\frac{R_s}{c}$ for $r_0 = 5R_s$!

Define characteristic time for collapse to B.H.:

$$\tau_{BH} \equiv \frac{R_s}{c} = 10^{-5}\,s\left(\frac{M}{M_\odot}\right)$$

Notice: observer moving with collapsing surface does NOT notice anything when crossing horizon (at R_s)

- The observer reaches the singularity in finite time

Story told by observer 2

- again, "radial plunge", but now we want
$r = r(t)$
↑ coordinate time = time measured by far-away observer

3 equations:

(i) $$\frac{dr}{d\tau} = -c\left[\frac{R_s}{r} - \frac{R_s}{r_0}\right]$$

(ii) $$e = c^2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau}$$

(iii) $$e = c^2\sqrt{1 - \frac{R_s}{r_0}}$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{-c}{\sqrt{\frac{r_0}{R_s} - 1}}\sqrt{\frac{r_0}{r} - 1}\left(1 - \frac{R_s}{r}\right)}$$

- Find approximate solution close to horizon
$(r \to R_s)$

$$\frac{dr}{dt} \simeq -c\left(1 - \frac{R_s}{r}\right) = \frac{-c}{r}\left(r - R_s\right)$$

$$\simeq \frac{-c}{R_s}\left(r - R_s\right)$$

→ Separation of variables:

$$\frac{dr}{(r - R_s)} = \frac{-c}{R_s}\, dt$$

$$\Rightarrow r(t) = R_s + b \exp\left[\frac{-c}{R_s} t\right],$$

where b = a constant.

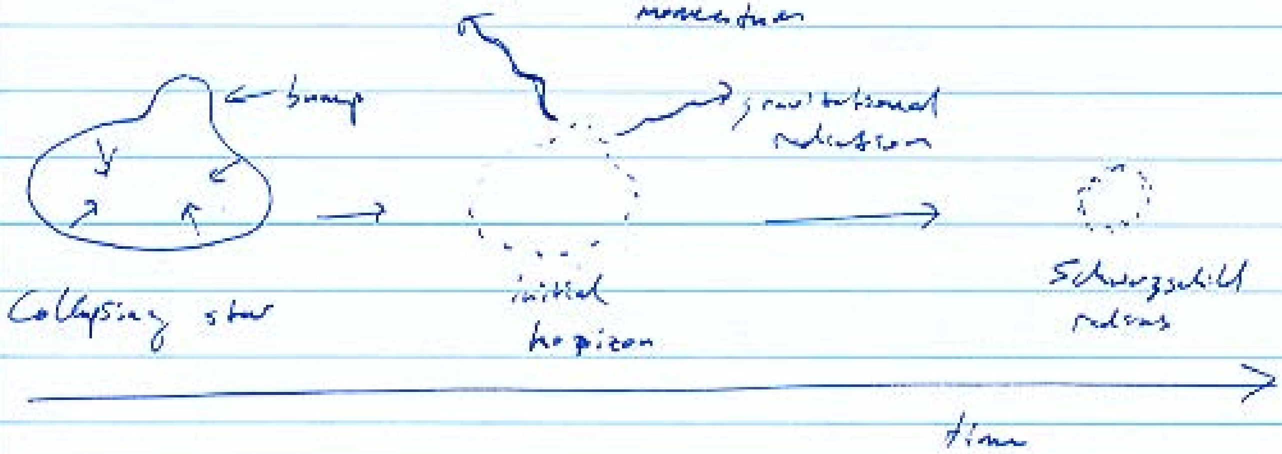
Notice: from viewpoint of far-away observer, radius approaches horizon asymptotically
→ Will take infinite time ($t \to \infty$) to reach R_s

- collapse appears to 'freeze' at horizon

• More general Black Holes

- BH, in general, are described by
3 parameters : M , j , Q

mass — angular momentum — electric charge



All properties "forgotten," except for M, j, Q :

$\boxed{\text{"BH have no hair"}}$