Curvature

- How curved is it?
  - The radius of an osculating circle can be used to measure curvature of a line at a given point.
  - Curvature = 1/(curvature radius)
    - Curvature is in units of 1/length
  - The signs posted on the road saying “R=300ft” or “R=500ft”
    - Which one is more curved?
  - A straight line (zero curvature) has R=∞

Gaussian Curvature

- Curvature of a surface
  - Draw two principal osculating circles at a given point on the surface
  - Obtain two principal curvature radii, $R_1$ and $R_2$
  - Gaussian curvature is given by $1/(R_1 R_2)$, up to the overall sign.
- $K$=Gaussian curvature
  - $K$ is in units of 1/area

Flat, Spherical, Hyperbolic

- Homogeneous and Isotropic space can be either flat, spherical, or hyperbolic.
  - $K$ is the same everywhere
- Flat
  - Zero $K$
    - $K=0$
  - Osculating circles on the same side
  - $R_1 = R_2 = \infty$
- Spherical
  - Positive $K$
    - $K=1/R^2 > 0$
  - Osculating circles on the same side
  - $R_1 = R_2 = R$
- Hyperbolic
  - Negative $K$
    - $K=-1/R^2 < 0$
  - Osculating circles on opposite sides
  - $R_1 = R_2 = R$

Measuring Curvature

- $\theta$=Sum of the angles of a triangle minus $\pi$
  - $\theta=K \times$ (area of triangle)
  - $\theta=0$ (flat)
  - $\theta>0$ (spherical)
  - $\theta<0$ (hyperbolic)
- $\theta$=Change in direction of an arrow through a closed circuit of the “vector transport”
  - $\theta=K \times$ (area enclosed by circuit)
  - This is neat – try it yourself!
Distances in Curved Space

- In flat space (Euclidean space), a distance between two points on a surface is given by
  \((\text{Distance})^2 = (x\text{-int.})^2 + (y\text{-int.})^2\)
  \(ds^2 = dx^2 + dy^2\)
- In curved space (non-Euclidean), Euclidean formula no longer applies:
  \((\text{Distance})^2 = F(x\text{-int.})^2 + 2G(x\text{-int.})(y\text{-int.}) + H(y\text{-int.})^2\)
  \(ds^2 = Fdx^2 + 2Gdxdy + Hdy^2\)
  \(\text{F, G, H : Metric Coefficients}\)
  - Metric coefficients generally depend on \(x\) and \(y\)

Gauss’s Theorema Egregium

- Gaussian curvature is an “intrinsic” one.
  -- We don’t need to know anything about 3rd dimension to measure Gaussian curvature of 2-dimensional space
  * “Flatlanders” can measure curvature of their world (which is 2 dimensional) without knowing anything about the 3rd dimension.
  -- Therefore, we can measure Gaussian curvature of 3-dimensional space without knowing anything about the 4th dimension.
- **Theorema Egregium**
  -- How metric coefficients vary from point to point on a surface contains all the information of the geometry of the surface.

Riemannian Geometry

- Riemann has extended Gauss’s work to four and higher dimensions.
- Metric coefficients have:
  - 3 components in 2d
  - 6 components in 3d
  - 10 components in 10d
  - etc…
- **Riemann curvature:** generalization of Gaussian curvature in higher dimensions.
- Einstein used Riemannian geometry to construct the general theory of relativity.

Spacetime Metric of the Universe

\[ ds^2 = c^2 dt^2 - \left( \frac{dR^2}{1 - KR^2} + R^2 (d\alpha^2 + \sin^2 \alpha d\theta^2) \right) \]

- Important question: **What is \(K\)** of the universe?
  -- \(K\) determines curvature of 3-d space in which we are living
- According to Gauss’s theorema egregium, we can measure \(K\), without knowing anything about the 4th dimension.
  -- This implies that the shape of the observable universe can be determined!