On the effect of a starspot on the modes of oscillation of a toy Ap star model

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1. Introduction

Our aim here is to go some way towards obtaining a simple approximation to the eigenfunctions of oscillation of an Ap star. To this end, we have developed and tested in a previous study an approximation to the eigenfunctions of acoustic oscillation of a toy model: namely, a cuboidal cavity within which the sound speed c is constant and on one of whose boundaries conditions vary with position. Specifically, the cavity occupied $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq h$, and the oscillation pressure (or density) eigenfunction $\Psi(x, y, z; \omega)$ of a mode with frequency ω satisfied

$$\nabla^2\Psi + k^2\Psi = 0 \tag{1}$$

in the cavity, where $k \equiv \omega/c$, subject to

$$\begin{aligned} \partial_x\Psi &= 0, && x = 0,\, a, \\ \partial_y\Psi &= 0, && y = 0,\, b, \\ \partial_z\Psi &= \begin{cases} 0, \\ -\lambda\, h^{-1} f(x,y)\,\Psi, \end{cases} && \begin{aligned} z &= 0, \\ z &= h. \end{aligned} \end{aligned} \tag{2}$$

In this model x plays the role of the colatitude, y of longitude, and z of radius. The boundary condition on $z = h$ was intended to mimic the direct effect of Lorentz forces in a thin surface boundary layer produced by disturbing a large-scale magnetic field by the oscillations. Accordingly, the function $f(x, y)$ was considered to vary only slowly with x and y. Our interest was, as it is here, in modes of high order n and low degree ℓ. Moreover, in practice we restricted attention, as we also do here, to y-independent functions f, modelling axisymmetric boundary variations in the star. Under these conditions we were able to justify approximating Ψ by the function

$$\hat{\Psi}(x, z) = \Lambda(x, y)\cos(k_z z), \tag{3}$$

where $k_z(x) = (n\pi + \Delta\delta)/h$, in which $\Delta\delta(x)$ is chosen to ensure that the boundary condition on $z = h$ is satisfied; this requires that $(n\pi + \Delta\delta)\tan(\Delta\delta) = \lambda f$. Because the background state is independent of y, we may seek solutions proportional to $e^{\mathrm{i}my}$, i.e., $\Lambda(x, y) = \Lambda(x)e^{\mathrm{i}my}$. On substituting the representation (3) into equation (1) and neglecting appropriate small terms, we then obtained

$$\Lambda'' + K\Lambda = 0, \tag{4}$$

where a prime denotes differentiation with respect to x, and where

$$K(x) = k^2 - m^2\pi^2/b^2 - (n\pi + \Delta\delta)^2/h^2. \tag{5}$$

Equations (4) and (5), subject to the boundary conditions $\Lambda' = 0$ on $x = 0$ and $x = a$, constitute an eigenvalue problem for k^2. We have demonstrated that, except under certain rare circumstances in which the frequencies of two modes nearly coincide, the eigenvalues k^2 are very close to the eigenvalues of the exact problem posed by equation (1) and conditions (2), and that the resulting representation $\hat{\Psi}$ is close to Ψ.

The question we pose here is: can such a procedure be used to represent the oscillations of stars with spots? In other words how well does the solution to the system (3)–(5) approximate the solution to the exact problem when $f(x)$ no longer varies slowly? Our estimates of the magnitudes of the terms that we neglected in obtaining equation (4) are small compared with the terms that we retained only when $|af'/f| = \mathrm{O}(1)$. What transpires if that condition is not satisfied? In this report we simply adopt our approximation procedure to describe a spotted model without prior justification, and compare the solution with the exact result.

2. Exact Solution

Once the function $f(x)$ in equation (2) is specified, it is straightforward to obtain an exact solution of the problem (Montgomery & Gough, in preparation). To simulate a star spot with a sharp boundary, we choose for the present case the discontinuous function

$$f(x) = \begin{cases} 0, & x < 0.43\,a, \\ 1, & x > 0.43\,a. \end{cases} \tag{6}$$

If we specialize to the two-dimensional case which is independent of y $(m = 0)$, then the exact solution takes the form

$$\Psi(x, z) = \sum_{\ell'=0}^{\infty} A_{\ell'} \cos(\ell'\pi x/a) \begin{cases} \cos(\kappa_{\ell'} z/h)\ , & \ell' < k\,a/\pi, \\ \cosh(\kappa_{\ell'} z/h)\ , & \ell' > k\,a/\pi, \end{cases} \tag{7}$$

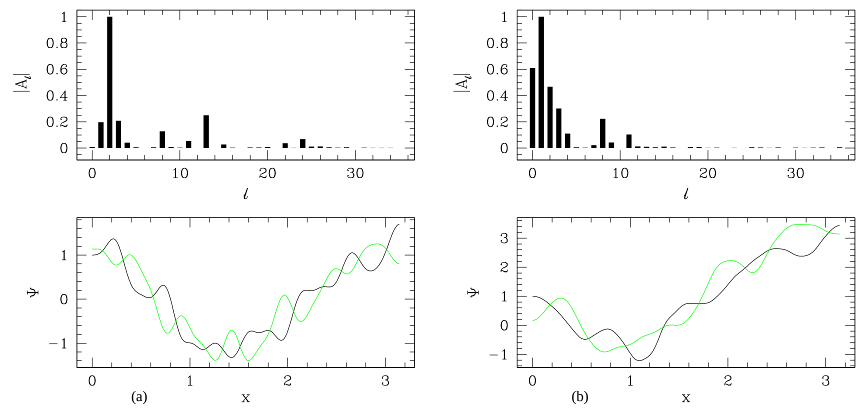

Figure 1. (a) Upper panel: Magnitudes $|A_\ell|$ of the coefficients in equation (7). The parameters assumed are $a = \pi$, $h = 2^{-1/4}\pi$, and $\lambda = 10.0$. In the $\lambda = 0$ limit, this mode corresponded to an $\ell = 2$, $n = 21$ mode. Lower panel: the eigenfunction Ψ evaluated on the upper boundary, $z = h$ (black line), and $-\Psi$ on the lower boundary $z = 0$ (green line). (b) The same as (a) but for $\lambda = 40.0$.

where

$$\kappa_{\ell'} \equiv \left| k^2 - (\ell'\pi/a)^2 \right|^{1/2} h. \tag{8}$$

In Figs. 1a and b, we show a particular solution corresponding to an $\ell = 2$, $n = 21$ mode. We see that a non-zero λ leads to the appearance of terms in addition to the original A_2 term. In fact, for $\lambda = 40.0$, the A_1 term is now the largest.

3. Comparison of Approximate and Exact Solutions

Fig. 2a shows a comparison of the exact and approximate solutions, for the solution shown in Fig. 1a. Each panel corresponds to a horizontal cross section through the box at a given height z, as indicated on each panel.

We note that since we have taken $f(x)$ to be discontinuous, then, by construction, our approximate solution is also discontinuous. This is seen most clearly in the panel labelled $z = 1.574$, in which there is a sharp kink at $x \sim 1.4$. Even in this case, however, we see that the approximate solution reproduces quite well the long-wavelength behaviour of the exact solution. In the comparisons in the other panels we see that the approximate solution is even better at reproducing the smoothed behaviour of the exact solution.

In an attempt to push the limits of our method, we have repeated the above analysis with a larger perturbation, having $\lambda = 40.0$. The exact solution is shown in Fig. 1b, and the comparison of the exact and

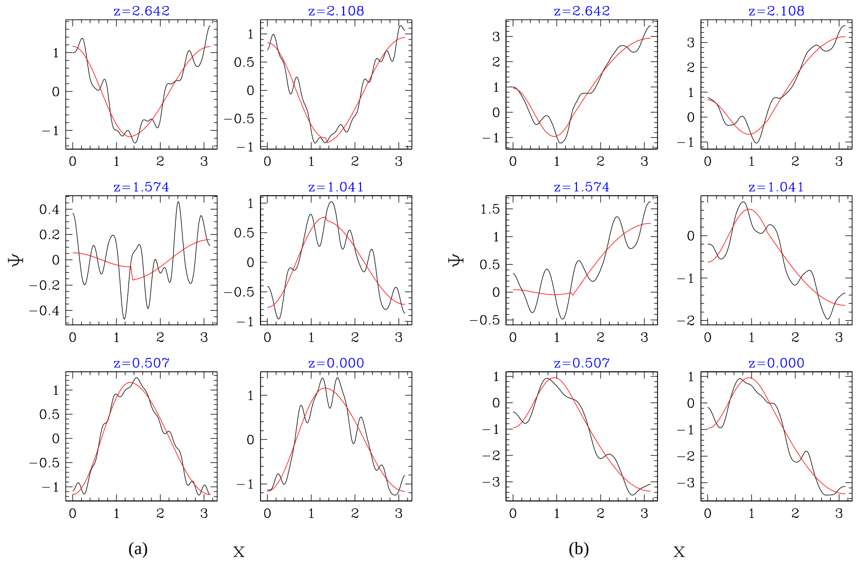

Figure 2. (a) A comparison of the exact solution in Fig. 1a (black line) and the approximate solution of equation 3 (red line) in different horizontal sections, each labelled with the value of z. (b) The same as (a) but for the case shown in Fig. 1b.

approximate solutions is shown in Fig. 2b. As before, we see that the approximate solution reproduces quite well the smoothed behaviour of the exact solution, even though in this case the eigenfunctions have been significantly altered from the unperturbed case.

This is a very important point, since observations of pulsating stars give us disk averages of the perturbed quantities (e.g., temperature, luminosity). Thus, the measurements themselves will filter out the high-frequency spatial components, leaving behind only the smoothly varying signal, which our approximation scheme can calculate very accurately.

4. Conclusions

The approximation method which we originally developed for treating a smooth perturbation to the upper boundary condition in a box was tested for the case in which there was a discontinuity in this boundary condition, designed to mimic the effect of a star spot. In spite of this, we found that our approximate solutions reproduced very well the smooth, long-wavelength behaviour of the exact solutions. Since disk-averaged observations will be sensitive mainly to the smooth component of the signal, our approximate eigenfunctions should provide a good description of the stellar eigenfunctions.