A Review of the Mathematics Used In AST 301

1 Units

If you say that a car is traveling at a speed of 70, most people in the United States will assume you mean 70 miles per hour. In Europe or Mexico, though, they will assume you mean 70 kilometers per hour. To avoid confusion the units of measurement, such as "miles per hour", must always be specified. Miles per hour is usually written

$$\text{miles/hour} \quad \text{or} \quad \frac{\text{miles}}{\text{hour}},$$

which means miles divided by hours and is exactly how speed is measured. Units are like numbers in the sense that they cancel. For example, a car traveling at 70 miles/hour for 3 hours, will go a distance

$$70\,\frac{\text{miles}}{\text{hour}} \times 3 \text{ hours} \;=\; 210 \text{ miles}$$

The "hours" cancel, leaving "miles". The units do not cancel unless the units are the same. If someone runs at a speed of 12 feet/second for 1 minute, it is not correct to calculate the distance traveled from 12 feet/sec × 1 minute because seconds and minutes do not cancel. One minute must first be converted to 60 seconds:

$$\textit{Wrong:} \qquad 12\,\frac{\text{feet}}{\text{second}} \times 1 \text{ minute} \;=\; ?$$

$$\text{Right:} \qquad 12\,\frac{\text{feet}}{\text{second}} \times 60 \text{ seconds} \;=\; 720 \text{ feet}.$$

2 Metric Units

Astronomers use the metric system. As its name implies, it is based on the meter, a unit of length that equals about 40 inches. The meter can be subdivided into one hundred centimeters or into one thousand millimeters, or they can be taken in groups of one thousand meters to form a kilometer:

$$\begin{aligned} 1 \text{ m} &= \sim 40 \text{ inches} \\ 1 \text{ m} &= 100 \text{ cm} = 1000 \text{ mm} \\ 1000 \text{ m} &= 1 \text{ km} \end{aligned}$$

The unit of mass in the metric system is the gram. The kilogram is 1000 gm (roughly 2.2 pounds).

3 Scientific Notation for Numbers

The numbers used by astronomers can be cumbersome because they are often very large or very small. Scientific notation makes large numbers much easier to read, write, and use. For example, the mass of our sun is

2,000,000,000,000,000,000,000,000,000,000,000 grams

(2 followed by 33 zeros). In scientific notation the sun's mass is 2×10^{33} grams (pronounced "two times ten to the thirty-three"). Positive powers represent large numbers and negative powers represent small numbers:

$$\begin{aligned} 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \end{aligned}$$

Remember this by writing down 1, and then moving the decimal point two places to the left to get 10^{-2} or two places to the right to get 10^2 and so on.

Consider the product $1000 \times 1,000,000$. Written the long way, the product is

$$1000 \times 1,000,000 \;=\; 1,000,000,000$$

In powers of ten notation, the product is written

$$10^3 \times 10^6 \;=\; 10^9$$

Note that $3 + 6 = 9$, a specific example of the general rule for multiplying numbers written in scientific notation:

• To multiply, add the powers of ten. •

To divide 10,000,000 by 1000 the long way, one takes

$$10,000,000 \;/\; 1000 \;=\; 10,000$$

In scientific notation this becomes

$$10^7 \;/\; 10^3 \;=\; 10^4$$

Note that $7 - 3 = 4$, an example of the general rule for dividing numbers written in scientific notation:

• To divide, subtract the powers of ten. •

Here is an example involving negative numbers

$$10^5 \;/\; 10^{19} \;=\; 10^{-14}$$

Notice that $5 - 19 = -14$. The inverse of a number written in scientific notation is just a special example of division:

$$1 \ / \ 10^5 \ = \ 10^0 \ / \ 10^5 \ = \ 10^{-5}$$

The last step comes from $0 - 5 = -5$.

Here is how scientific notation handles numbers that are not exact powers of ten:

$$300000 \ = \ 3 \times 100000 \ = \ 3 \times 10^5$$

or

$$6,5384 \ = \ 6.5384 \times 10,000 \ = \ 6.5384 \times 10^4$$

or

$$0.0483 \ = \ 4.83 \times 0.01 \ = \ 4.83 \times 10^{-2}$$

The part of the number before the $\times$ sign usually lies between 1 and 10 in scientific notation, but this is just a convention. The number 0.0483 could also be written 0.483×10^{-1} or even 0.000483×10^2. To multiply these more complicated numbers in scientific notation,

• multiply the two parts of complicated numbers separately. •

For example,

$$\begin{aligned}(6 \times 10^{19}) \times (4 \times 10^7) \ &= \ (6 \times 4) \times (10^{19} \times 10^7) \ = \ 24 \times 10^{26} \\ &= \ 2.4 \times 10^{27},\end{aligned}$$

and to divide more complicated numbers, divide the two parts separately:

$$\begin{aligned}7 \times 10^{24} \ / \ 2 \times 10^{36} \ &= \ (7 \ / \ 2) \times (10^{24} \ / \ 10^{36}) \\ &= \ 3.5 \times 10^{-12}\end{aligned}$$

To add or subtract numbers in scientific notation, both numbers must have the same power of ten. For example,

$$\begin{aligned}4 \times 10^8 \ + \ 2 \times 10^6 \ &= \ 400 \times 10^6 \ + \ 2 \times 10^6 \ = \ 402 \times 10^6 \\ &= \ 4.02 \times 10^8\end{aligned}$$

4 Other Powers

Consider the number 3^2. This means "three squared" or "three multiplied by itself twice", so that $3^2 = 3 \times 3 = 9$. Generalizing from this example, we have

$$\begin{aligned}3^3 \ &= \ 27 \\ 3^2 \ &= \ 9 \\ 3^1 \ &= \ 3 \\ 3^0 \ &= \ 1\end{aligned}$$

To calculate negative powers, use inverses:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

The power does not have to be a whole number. Consider the number $9^{1/2}$. This is the number, which, when multiplied by itself, gives 9 because

$$9^{1/2} \times 9^{1/2} = 9^{(1/2+1/2)} = 9^1 = 9$$

So, $9^{1/2}$ is the square root of 9:

$$9^{1/2} = 3$$

By the same logic raising a number to the 1/3 power means the cube root. For example,

$$27^{1/3} = 3.$$

5 Equations

All equations have an equals sign (the "=") and something on both sides of the equals sign. For example,

$$x = y.$$

Equations mean that both sides of the equals sign have the same value, in this case x and y have the same value. The single most important rule for manipulating equations is

- Do the same thing to both sides of the equation so the equality is preserved. •

For example, one can add 5 to both sides of the equation:

$$x + 5 = y + 5$$

and the equality is preserved; or multiply both sides of the equation by 3:

$$3x = 3y,$$

and the equality is preserved; or even square both sides of the equation:

$$x^2 = y^2,$$

and the equality is preserved. The term "solve the equation for x" means "manipulate the equation so that x is alone on the left side of the equation." Here is how to solve the equation $x + 5 = 13$:

$$\begin{array}{rrcl} \text{begin with} & x+5 & = & 13 \\ \text{subtract 5} & x+5-5 & = & 13-5 \\ \text{the solution} & x & = & 8 \end{array}$$

It may take many steps to solve more complicated equations. Here is how to solve the equation $3(x^2+4) = 87$ for x:

$$
\begin{array}{rrcl}
\text{begin with} & 3(x^2+4) & = & 87 \\
\text{divide by 3} & \dfrac{3(x^2+4)}{3} & = & \dfrac{87}{3} \\
 & x^2+4 & = & 29 \\
\text{subtract 4} & x^2+4-4 & = & 29-4 \\
 & x^2 & = & 25 \\
\text{take the square root} & x & = & 5 \ \text{ or } \ -5
\end{array}
$$

Note that the square root of 25 can be either 5 or -5.

Equations can be re-arranged before numbers are inserted to make them more suitable for a particular use. For example, the relation between speed, distance, and time is:

$$
\begin{aligned}
\text{speed} \times \text{time} &= \text{distance.} \\
s \times t &= d
\end{aligned}
$$

This equation can be rearranged to give an equation for speed by dividing each side by time:

$$
\begin{aligned}
\frac{s \times t}{t} &= \frac{d}{t} \\
s &= \frac{d}{t}.
\end{aligned}
$$

6 Angles

A circle is divided into 360 degrees, written 360°. The degree is further divided into 60 minutes. To distinguish these minutes from minutes of time, we write either

$$1^\circ = 60 \text{ arc minutes}$$

or

$$1^\circ = 60'$$

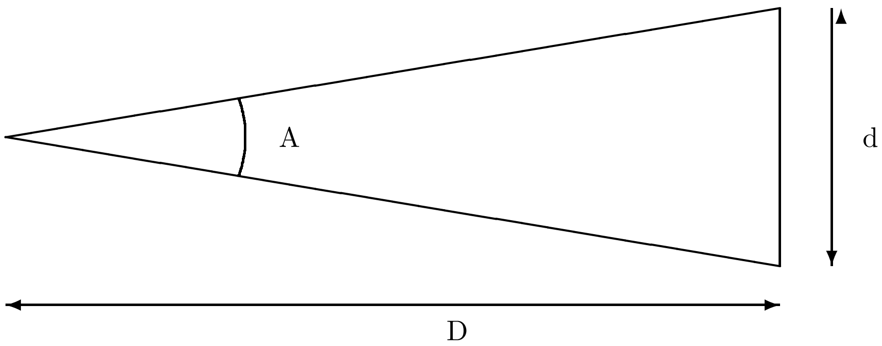
The minute of arc is yet further divided into 60 seconds of arc, written either

$$1' = 60 \text{ arc seconds}$$

or

$$1' = 60''$$

Many of the angles used in astronomy are very small, among them the parallaxes of stars. In the pictured triangle, d is the base of the triangle, D is its height, and A is



the angle between the sides of the triangle. Calculations concerning triangles normally require trigonometric formulae, but if the triangle is skinny (if the angle A is small), there is a simple formula relating A, D, and d:

$$\mathrm{A} \;=\; 206,000\,\frac{\mathrm{d}}{\mathrm{D}} \;\text{ arc seconds}$$

where d and D can be any units, but the units must be the same for both.

As an example, the angular diameter of the moon is $1/2^\circ$ and its distance is 400,000 km. This means that if the moon is placed along the base of the triangle so that its diameter is d, and if its distance D is 400,000 km, then the angle A is equal to $1/2^\circ$. To find the true diameter of the moon, we set

$$\begin{aligned} \mathrm{A} &= 1/2^\circ \;=\; 1800'' \\ \mathrm{D} &= 400,000 \text{ km} \end{aligned}$$

and then, re-arranging the formula, we have

$$\begin{aligned} \mathrm{d} &= \frac{\mathrm{A}\times\mathrm{D}}{206,000} \\ &= \frac{1800\times 400,000}{206,000} \\ &= 3495 \text{ km.} \end{aligned}$$

7 Ratios

Ratios are often useful in astronomy. For example, the ratio of the sun's radius to the size of the solar system is the same as the ratio of the radius of an atom's nucleus to the atom's size. Ratios often behave like equations. If the ratio of x to 7 equals the ratio of 3 to 42, we can write

$$\frac{x}{5} \;=\; \frac{3}{45}$$

and this is an equation. The equation can be solved for x in the usual way. Multiply both sides of the equation by 5 to get

$$\begin{aligned} \frac{x}{5} \times 5 &= \frac{3}{45} \times 5 \\ x &= \frac{1}{3} \end{aligned}$$

Some laws of astronomy can be written as

$$\textit{(something)} = \textit{(constant)} \times \textit{(something else)}.$$

For example, the energy emitted by a star depends on the 4th power of the star's temperature:

$$L_* = constant \times T^4$$

where L_* means "the luminosity of a star" and T is its temperature. To calculate the luminosity of a star it is necessary to know the value of the constant, but by using ratios it is sometimes possible to do calculations without knowing the value of the constant. It is not, for example, necessary to know the constant to calculate the ratio of the luminosities of two stars with different temperatures. Suppose a star has twice the sun's temperature. If the sun's temperature is T and the star's temperature is $2T$, the luminosity of the star is

$$L_* = const \times (2T)^4,$$

and the luminosity of the sun is

$$L_\odot = const \times T^4,$$

where $L_\odot$ means "luminosity of the sun". The ratio of the two luminosities is

$$\begin{aligned} \frac{L_*}{L_\odot} &= \frac{const \times (2T)^4}{const \times T^4} \\ &= \frac{const \times 2T \times 2T \times 2T \times 2T}{const \times T \times T \times T \times T} \\ &= 2 \times 2 \times 2 \times 2 = 16. \end{aligned}$$

Notice that the constant and the sun's temperature cancel, so it was not necessary to know their values.

AST 301, Math Practice Homework

This homework is for your benefit alone. Do not hand it in, but if you have difficulty with any of the problems, come see us and we will help you. Remember: questions involving square roots generally have two answers.

1. Zippy the snail travels 6 inches in 1 minute. What is its speed in millimeters/second?
2. How many millimeters are there in 1 kilometer?
3. Express the following numbers in scientific notation.
   (a) 2137
   (b) 70000
   (c) 1/500
4. Write out the following numbers in full.
   (a) 3.44×10^5
   (b) 7×10^{-2}
   (c) 4×10^{-12}
5. Calculate the following quantities, and give the answers in scientific notation.
   (a) $10^2 \times 10^3 \times 10^6 \times 10^{-10} \times 10^3 \times 10^2$
   (b) $(3 \times 10^{-7}) \times (4 \times 10^8)$
   (c) $(6 \times 10^9) \div (3 \times 10^{10})$
6. What is 2^7 in "ordinary" numbers?
7. What is $81^{1/4}$?
8. What is the value of $3^4 \times 2^2$? What is its square root?
9. What is the square root of 1.6×10^7 (no calculators needed)?
10. Solve the following equations for x.
    (a) $7 + x = 19$
    (b) $3x - 4 = 8$
    (c) $(x - 4)^2 = 25$
    (d) $5 + x = 19 - x$
11. The sun has a diameter of 1.4×10^{11} cm and is 1.5×10^{13} cm from the Earth. What is the angular diameter of the sun in arc seconds? In degrees?