GRAVITATION AS AN EFFECTIVE FIELD THEORY

With an ultraviolet cut-off Λ,

$$I_\Lambda[g] = -\int d^4x \sqrt{-\text{Det}g}\Bigg[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda) R + g_{2a}(\Lambda) R^2 + g_{2b}(\Lambda) R^{\mu\nu} R_{\mu\nu} + \Lambda^{-2} g_{3a}(\Lambda) R^3 + \Lambda^{-2} g_{3b}(\Lambda) R R^{\mu\nu} R_{\mu\nu} + \ldots\Bigg].$$

The Λ-dependence of the dimensionless couplings $g_n(\Lambda)$ is such that physics is independent of Λ.

$$\Lambda \frac{d}{d\Lambda} g_n(\Lambda) = \beta_n\Big(g(\Lambda)\Big)$$

ASYMPTOTIC SAFETY

The theory is safe from couplings blowing up as Λ increases if $\beta(g_*) = 0$ and $g(\Lambda)$ is on a trajectory attracted to g_*. (SW, 1976)

Trajectories with $g \to g_*$ for $\Lambda \to \infty$ form the *ultraviolet critical surface.* The physical requirement that actual couplings lie on UV critical surface may play the same role for theories including gravitation as does renormalizability in QED or QCD.

The number of free parameters in the theory equals the dimensionality of the UV critical surface.

Indications of Asymptotically Safe Gravitation

- Dimensional Continuation ($d = 2+\epsilon$)
  - SW 1979
  - Kawai, Kitazawa, & Ninomiya, 1993, 1996
  - Aida & Kitazawa, 1997 (2 loops)
  - Niedermaier 2003
- $1/N$ Expansion
  - Smolin 1982 ($R + C^2$)
  - Percacci, 2006
- Lattice Quantization
  - Ambjørn, Jurkewicz, & Loll, 2004, 2005, 2006, 2008

- Truncated 'Exact" Renormalization Group
  - Wegener & Houghton, 1973
  - Polchinski, 1984
  - Wetterich, 1993

  (Exact renormalization group equations link all $g_n(\Lambda)$. Truncate equations by setting all but a finite number of $g_n(\Lambda)$ equal zero, ignore the non-zero value of the other $\beta_n(g)$.)

  - Reuter, 1998
  - Dou & Percacci, 1998
    (gravity + free matter)
  - Souma, 1999 $(R + \lambda)$
  - Lauscher & Reuter, 2001 $(R + \lambda)$
  - Reuter & Saueressig, 2002 $(R+\lambda)$
  - Lauscher & Reuter, 2002
    $(R + \lambda + R^2)$

– Reuter & Saueressig, 2002
– Percacci & Perini, 2002, 2003
(constraints on free matter)
– Perini, 2004
– Litim, 2004
– Codello & Percacci, 2006
– Reuter & Saueressig, 2007
– Machado & Saueressig, 2007
– Litim, 2008

With only 2 non-zero couplings, UV critical surface is 2 dimensional.
With only 3 non-zero couplings, UV critical surface is 3 dimensional.
This was not encouraging.

Good News!

- Codello, Percacci, & Rahmede, 2008 (R^n, $0 \leq n \leq 6$)
- Codello, Percacci, & Rahmede, 2008 (R^n & matter, $0 \leq n \leq 6$)
- Benedetti, Machado, & Saueressig, 2009 ($R + \lambda + R^2 + C^2$)

All find a 3-dimensional UV critical surface.

COSMOLOGICAL APPLICATIONS

For a Robertson–Walker line element

$$d\tau^2 = dt^2 - a^2(t)\, d\vec{x}^2 \ .$$

the action takes the form

$$I_\Lambda[g_{\rm RW}] = V\Lambda^4 \int dt\; a^3(t)\, \mathcal{F}_\Lambda\Big(H(t), \dot{H}(t), \ldots\Big) \ ,$$

where

$$H(t) \equiv \dot{a}(t)/a(t)$$

Gravitational Field Equation:

$$\begin{aligned} 0 \;&= \mathcal{N}_\Lambda(H, \dot{H}, \ddot{H}, \ldots) \\ &\equiv \mathcal{F}_\Lambda - H\frac{\partial \mathcal{F}_\Lambda}{\partial H} + (-\dot{H} + 3H^2)\frac{\partial \mathcal{F}_\Lambda}{\partial \dot{H}} \\ &\qquad + H\frac{d}{dt}\left(\frac{\partial \mathcal{F}_\Lambda}{\partial \dot{H}}\right) + \ldots \end{aligned}$$

$$I_\Lambda[g] = -\int d^4x \sqrt{-\mathrm{Det}g}\Big[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda) R$$
$$+ g_{2a}(\Lambda) R^2 + g_{2b}(\Lambda) R^{\mu\nu} R_{\mu\nu}$$
$$+ \Lambda^{-2} g_{3a}(\Lambda) R^3 + \Lambda^{-2} g_{3b}(\Lambda) R R^{\mu\nu} R_{\mu\nu} + \ldots \Big]$$

gives the field equation

$$0 = \mathcal{N}_\Lambda(H, \dot{H}, \ddot{H}, \ldots)$$
$$= -g_0(\Lambda) + 6\Lambda^{-2} g_1(\Lambda) H^2$$
$$-\Lambda^{-4} g_{2b}(\Lambda)\Big(72 H^2 \dot{H} - 12 \dot{H}^2 + 24 H \ddot{H}\Big)$$
$$+\Lambda^{-6} g_{3a}(\Lambda)\Big(- 864\, H^6 + 7776 H^4 \dot{H}$$
$$+3240 H^2 \dot{H}^2 - 432\, \dot{H}^3 + 216\, H \ddot{H} (12 H^2 + 6 \dot{H})\Big)$$
$$+\Lambda^{-6} g_{3b}(\Lambda)\Big(- 216 H^6 + 2160 H^4 \dot{H} + 1008\, H^2 \dot{H}^2$$
$$-144 \dot{H}^3 + H \ddot{H} (720 H^2 + 432 \dot{H})\Big) + \ldots .$$

De Sitter Solution

$$a(t) \propto \exp(\bar{H}t) \;, \quad \bar{H} \text{ constant}$$

where

$$\begin{aligned} 0 = & \; N_\Lambda(\bar{H}) \equiv \mathcal{N}_\Lambda(\bar{H}, 0, 0, \ldots) \\ = & \; -g_0(\Lambda) + 6\, g_1(\Lambda)\, (\bar{H}/\Lambda)^2 - 864\, g_{3a}(\Lambda)\, (\bar{H}/\Lambda)^6 \\ & -216\, g_{3b}(\Lambda)\, (\bar{H}/\Lambda)^6 + \ldots \end{aligned}$$

A hard choice:

- If $\Lambda \ll \bar{H}$ then radiative corrections are small, but series diverges badly.
- If $\Lambda \gg \bar{H}$ then series may be dominated by low terms, but radiative corrections are important, as shown by Λ dependence of $\bar{H}$. (Near fixed point, $\bar{H} \propto \Lambda$.)

Choose $\Lambda \approx \bar{H}$, but at a local minimum of radiative corrections. Write true expansion rate as

$$H_{\text{true}} = \bar{H}(\Lambda) + \Delta H(\Lambda)$$

Choose optimal cutoff Λ so

$$\frac{d}{d\Lambda}\Delta H(\Lambda) = 0$$

But H_{true} is independent of Λ, so at optimal cutoff

$$\frac{d}{d\Lambda}\bar{H}(\Lambda) = 0$$

By definition, for all Λ,

$$N_\Lambda\Big(\bar{H}(\Lambda)\Big) = 0$$

so at optimal cutoff

$$\frac{\partial}{\partial\Lambda}N_\Lambda\Big(\bar{H}\Big) = 0$$

.

This gives two equations for $\bar{H}$ & Λ:

$$0 = -g_0(\Lambda) + 6\, g_1(\Lambda)\, (\bar{H}/\Lambda)^2 - 864\, g_{3a}(\Lambda)\, (\bar{H}/\Lambda)^6 \\ -216\, g_{3b}(\Lambda)\, (\bar{H}/\Lambda)^6 + \dots$$

and

$$0 = -12 \left(\frac{\bar{H}}{\Lambda}\right)^2 g_1(\Lambda) + 5184 \left(\frac{\bar{H}}{\Lambda}\right)^6 g_{3a}(\Lambda) \\ +1296 \left(\frac{\bar{H}}{\Lambda}\right)^6 g_{3b}(\Lambda) + \dots \\ -\beta_0\Big(g(\Lambda)\Big) + 6\, \beta_1\Big(g(\Lambda)\Big)\, (\bar{H}/\Lambda)^2 \\ -864\, \beta_{3a}\Big(g(\Lambda)\Big)\, (\bar{H}/\Lambda)^6 \\ -216\, \beta_{3b}\Big(g(\Lambda)\Big)\, (\bar{H}/\Lambda)^6 + \dots\,.$$

We expect solutions with

$$\bar{H} \approx \Lambda \approx M$$

($M \equiv$ scale at which $g_n(\Lambda)$ approach fixed point.)

$$g_n(\Lambda) \to g_{n*} + \sum_i u_{in} \left(\frac{\Lambda}{M}\right)^{\lambda_i}$$

where

$$\sum_m B_{nm}\, u_{im} = \lambda_i\, u_{in}\,, \quad B_{nm} \equiv \left(\frac{\partial \beta_n(g)}{\partial g_m}\right)_*$$

and

$$\max(u_{in}) = O(1)$$

Free parameters are M and relative normalization of eigenvectors u_{in}

$$H(t) = \bar{H} + \delta H(t)$$

$$c_0(\bar{H},\Lambda)\frac{\delta H}{\bar{H}} + c_1(\bar{H},\Lambda)\frac{\delta \dot{H}}{\bar{H}^2} + c_2(\bar{H},\Lambda)\frac{\delta \ddot{H}}{\bar{H}^3} + \ldots = 0$$

$$c_0(\bar{H},\Lambda) = 12\left(\frac{\bar{H}}{\Lambda}\right)^2 g_1(\Lambda)$$

$$-5184\left(\frac{\bar{H}}{\Lambda}\right)^6 g_{3a}(\Lambda) - 1296\left(\frac{\bar{H}}{\Lambda}\right)^6 g_{3b}(\Lambda) + \ldots\ ,$$

$$c_1(\bar{H},\Lambda) = -216\, g_{2a}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^4 - 72\, g_{2b}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^4$$

$$+7776\, g_{3a}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^6 + 2160\, g_{3b}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^6 + \ldots\ ,$$

$$c_2(\bar{H},\Lambda) = -72\, g_{2a}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^4 - 24\, g_{2b}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^4$$

$$+2592\, g_{3a}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^6 + 720\, g_{3b}(\Lambda)\left(\frac{\bar{H}}{\Lambda}\right)^6 + \ldots\ ,$$

Solution: $\delta H \propto \exp(\xi \bar{H} t)$, where

$$c_0(\bar{H}, \Lambda) + c_1(\bar{H}, \Lambda)\,\xi + c_2(\bar{H}, \Lambda)\,\xi^2 + \ldots = 0$$

For $\xi > 0$, number of e-foldings is $N \simeq 1/\xi$. If $|c_0(\Lambda)| \ll 1$ and $|c_n(\Lambda)| \approx 1$ for $n \geq 1$, then

$$N \simeq |c_1(\Lambda)/c_0(\Lambda)| \gg 1$$

At optimal cutoff

$$\begin{aligned} c_0(\Lambda) = & -\beta_0\Big(g(\Lambda)\Big) + 6\,\beta_1\Big(g(\Lambda)\Big)\,(\bar{H}/\Lambda)^2 \\ & -864\,\beta_{3a}\Big(g(\Lambda)\Big)\,(\bar{H}/\Lambda)^6 \\ & -216\,\beta_{3b}\Big(g(\Lambda)\Big)\,(\bar{H}/\Lambda)^6 + \ldots \end{aligned}$$

We get many e-foldings of inflation if couplings are near fixed point, where $\beta_n = 0$.