Horava-Lifshitz Theory of Gravity & Applications to Cosmology

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I. Introduction

A. GR is not renormalizable [S. Weinberg, in General Relativity, An Einstein Centenary Survey, 1980]:

- GR is described by the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g^{(4)}}R^{(4)},$$

which is Lorentz invariant:

$$t \to \xi^0(t,x), \quad x^i \to \xi^i(t,x).$$

- Weak-field approximations yield,

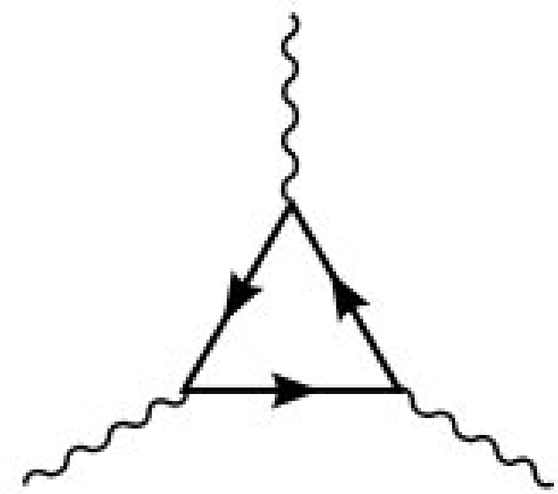
$$\begin{aligned} \Box R^{(4)}_{abcd} &\equiv \left(\partial_t^2 - \nabla^2\right) R^{(4)}_{abcd} = 0 \\ \Rightarrow \quad R^{(4)}_{abcd} &\propto e^{ik_\lambda x^\lambda},\ k_\lambda = (\omega, k_i)\,,\ \omega^2 = c^2\mathbf{k}^2. \end{aligned}$$

I. Introduction (Cont.)

- GR is not renormalizable, where the graviton propagator is given by

$$G(\omega, \mathbf{k}) = \frac{1}{k^2},$$

$k^2 \equiv \omega^2 - c^2\mathbf{k}^2$. In the Feynmann diagram, each internal line picks up a propagator, while each loop picks up an integral,



$$\int d\omega d^3k \propto k^4.$$

I. Introduction (Cont.)

- The total contribution from one loop and one internal propagator is

$$k^4 k^{-2} = k^2 \to \infty, \ \ \text{as} \ \ k \to \infty.$$

At increasing loop orders, the Feynman diagrams require counterterms of ever-increasing degree in curvature $\Rightarrow$ – Not renormalizable.

- Improved UV behavior can be obtained, if *Lorentz invariant* higher-derivative curvatures added. For example, when quadratic terms, $R^{(4)^2}$, are added,

$$\frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + ... = \frac{1}{k^2 - G_N k^4}.$$

I. Introduction (Cont.)

- At high energy, the propagator $\propto 1/k^4$, and the total contribution from one loop and one internal propagator now is

$$k^4 k^{-4} = k^0 \rightarrow \text{ finite, as } \quad k \rightarrow \infty.$$

– The problem of the UV divergency is cured. But **two new problems** occur,

$$\frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/G_N},$$

$\frac{1}{k^2}$: **spin-0 graviton;** $-\frac{1}{k^2 - 1/G_N}$: **ghost.**

I. Introduction (Cont.)

B. Scalar Field:

- A free scalar field in $d+1$ flat spacetime:

$$S_\phi = \int dt d^d x \left(\dot{\phi}^2 + \phi \nabla^2 \phi\right), \quad \Rightarrow \quad \Box \phi = 0,$$

which is Lorentz invariant:

$$t \rightarrow \xi^0(t, x), \quad x^i \rightarrow \xi^i(t, x).$$

The propagator of the scalar field is,

$$G^\phi(\omega, \mathbf{k}) = \frac{1}{k^2},$$

$\Rightarrow$ not renormalizable!!!

I. Introduction (Cont.)

- Lifshitz scalar & its power-counting Renormalizability [E.M. Lifshitz, Zh. Eksp. Teor. Fiz. 11, 255; 269 (1941); M. Visser, PRD80, 025011 (2009)]:

$$S_\phi = \int dt d^d x \left(\dot{\phi}^2 - \phi \left(-\nabla^2\right)^z \phi\right),$$

z: positive integer. **When $z > 1$, Lorentz invariance is broken,** and t and x have different dimensions,

$$[dt] = [dx]^z \equiv [k]^{-z}, \quad [\partial_t]^2 = \left[\nabla^2\right]^z = [k]^{2z}.$$

– **Anisotropic scaling.** Since $\left[S_\phi\right] = [1]$, we must have

$$[\phi] = [dx]^{(z-d)/2} = [k]^{-(z-d)/2} \quad \Rightarrow [\phi]|_{z=d} = [1].$$

I. Introduction (Cont.)

- The action is invariant under the re-scaling:

$$t \longrightarrow \ell^z t, \quad x \longrightarrow \ell x.$$

- Adding various sub-leading (**relevant**) terms, we have

$$S_\phi = \int dt d^d x \left\{ \dot{\phi}^2 - \phi \left[m^2 - c^2 \nabla^2 + ... + \left(-\nabla^2\right)^z \right] \phi \right\},$$

$\left(-\nabla^2\right)^z$: **marginal,** and

$$[c] = [dx/dt] = [dx]^{1-z} = [k]^{z-1} = [m]^{(z-1)/z}.$$

I. Introduction (Cont.)

- Adding interactions, we have

$$S_{\phi}^{Int.} = \int dtd^dxV(\phi) = \int dtd^dx\left(\sum_{n=1}^{N} g_n\phi^n\right),$$

where

$$[g_n] = [k]^{d+z-n(d-z)/2}, \;\; \Rightarrow \left[S_{\phi}^{Int.}\right] = [1].$$

- It can be shown that when $z \geq d$, the scalar becomes renormalizable!!!

II. Horava-Lifshitz theory [P. Horava, PRD79 (2009) 084008]:

- **The fundamental assumption:** Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at high energies.

- **Observations:** The action of the Lifshitz scalar field involves only quadratic term, $\dot{\phi}^2$, to avoid the ghost problem, but must contain higher-order spatial derivative terms, $\left(\nabla^2\phi\right)^z$, to improve the UV divergency.
  Theorefore, renormalization & non-ghost $\Rightarrow$ Only $\dot{g}_{\mu\nu}^2$ term appear in the action, but higher-order spatial derivative terms, $\left(\nabla^2 g_{\mu\nu}\right)^z$ with $z \geq d$, should be included.

II. Horava-Lifshitz theory (Cont.):

- **Horava's Assumption 1: Foliation-preserving Diffeomorphisms.** In addition to being a differentiable manifold, the spacetime $\mathcal{M}$ carries an extra structure – that of a codimension-one foliation, given by,

$$ds^2 = -N^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \quad (1)$$

$N(t), N^i(t,x), g_{ij}(t,x)$ are the fundamental quantities. This foliation is invariant only under,

$$t \rightarrow f(t), \quad x^i \rightarrow \xi^i(t,x). \quad (2)$$

Note (a): Comparing with these in GR, we lose one degree of freedom: $t \rightarrow f(t,x) \Rightarrow$ spin-0 gravity.

II. Horava-Lifshitz theory (Cont.):

- Note (b): With the foliation-preserving diffeomorphisms, both $K^{ij}K_{ij}$ and K become invariant, so the most general kinetic part of the action now is,

$$\mathcal{L}_K = g_K \left[K^{ij}K_{ij} - (1-\xi)K^2\right], \tag{3}$$

while the general diffeomorphisms of GR requires $\xi = 0$.

- Note (c): The recovery of GR at large distance (low energy) requires

$$g_K = \frac{1}{2\kappa^2}, \quad \xi \to 0.$$

II. Horava-Lifshitz theory (Cont.):

- **Horava's Assumption 2: Projectability condition.** The lapse function N is function of t only,

$$N = N(t). \tag{4}$$

Note that this is a **natural** but **not necessary** assumption, and is preserved by the foliation-preserving diffeomorphisms (2). However, to solve the inconsistence problem, it is found that it is needed [M. Li & Y. Pang, arXiv:0905.2751].

II. Horava-Lifshitz theory (Cont.):

- **Horava's Assumption 3:** **Detailed balance condition.**

  To construct the part that contains higher spatial derivatives, Horava assumed the detailed balance condition. With the choice $z = d$, it can be shown that this condition uniquely reduces the spatial higher-derivative (potential) part simply to

$$\mathcal{L}_V = g_V C_{ij} C^{ij}, \tag{5}$$

C_{ij}: the Cotton tensor,

$$C^{ij} = \epsilon^{ikl} \nabla_k \left(R^j_l - \frac{1}{4} R \delta^i_l \right).$$

II. Horava-Lifshitz theory (Cont.):

So far most of the works have been done without the projectability condition, but with the detailed balance.

The main reasons are:

- The detailed balance condition leads to a very simple action, and the resulted theory is much easy to deal with.
- Abandoning projectability condition gives rise to local rather than global Hamiltonian and energy constraints.

II. Horava-Lifshitz theory (Cont.):

Problems with detailed balance condition:

- A scalar field is not UV stable [G. Calcagni, arXiv:0905.3740].
- Gravitational scalar perturbations is not stable for any choice of $\xi(\lambda)$ [C. Bogdanos & E. Saridakis, arXiv:0907.1636].
- Requires a non-zero (negative) cosmological constant [P. Horava, PRD79 (2009) 084008].
- Strong coupling [H. Lu, J. Mei & C.N. Pope, PRL103, 091301 (2009); C. Charmousis et al, JHEP, 08, 070 (2009); S. Mukhyama, 0905.3563; Blas et al, arXiv:0906.3046; Koyama & Arroja, arXiv:0910.1998].
- Breaks the parity in the purely gravitational sector [T. Sotiriou et al, PRL102 (2009) 251601].

II. Horava-Lifshitz theory (Cont.):

Problems with detailed balance condition:

- Makes the perturbations not scale-invariant [X. Gao, et al, arXiv:0905.3821].

- ...

II. Horava-Lifshitz theory (Cont.):

Problems without projectability condition:

- The theory is usually not consistence, because the corresponding Poisson brackets of the Hamiltonian density do not form a closed structure [M.Li & Y. Pang, arXiv:0905.2751].
- ???

Problems with projectability condition:

- Gravitational scalar perturbations are either ghost or not stable [T.S. Sotirious et al, arXiv:0905.2798; AW & R. Maartens, arXiv:0907.1748].
- ???

Therefore, the one with projectability but without detailed balance condition seems to have more potential to solve the problems found so far.

II. Horava-Lifshitz theory (Cont.):

Sotiriou-Visser-Weinfurtner Generalization [PRL102 (2009) 251601]:

The most general generalization of the HL theory without detailed balance but with projectability condition [for $z = d = 3$],

$$
\begin{aligned}
S &= \int dt d^3x N\sqrt{g}\left[\frac{1}{2\kappa^2}\left(\mathcal{L}_K - \mathcal{L}_V\right) + \mathcal{L}_M\right],\\
\mathcal{L}_K &= K^{ij}K_{ij} - (1-\xi)K^2,\\
\mathcal{L}_V &= 2\Lambda - R + \frac{1}{\zeta^2}\left(g_2 R^2 + g_3 R_{ij}R^{ij}\right)\\
&\quad + \frac{1}{\zeta^4}\left(g_4 R^3 + g_5 R\, R_{ij}R^{ij} + g_6 R^i_j R^j_k R^k_i\right)\\
&\quad + \frac{1}{\zeta^4}\left[g_7 R\nabla^2 R + g_8\left(\nabla_i R_{jk}\right)\left(\nabla^i R^{jk}\right)\right],
\end{aligned}
\tag{6}
$$

$\mathcal{L}_M$: the matter part; g_n: dimensionless coupling constants; $\zeta^2 = 1/(2\kappa^2) = (16\pi G)^{-1}$.

II. Horava-Lifshitz theory (Cont.):

- ξ: a dynamical coupling constant, susceptible to quantum corrections, and is expected to flow from its UV value, ξ_{UV}, to its IR one, $\xi = 0$, whereby GR is recovered.

- $g_2, g_3, ..., g_8$: all related to the breaking of Lorentz invariance, and highly suppressed by the Planck scale in the IR limit.

III. Cosmology in Horava-Lifshitz theory:

A. Spin-0 Graviton [AW & R. Maartens, arXiv:0907.1748]:

- Linear Perturbations in Minkowski background:

$$ds^2 = -(1+2\phi)dt^2 + 2B_{,i}dx^i dt + \left[(1-2\psi)\delta_{ij} + 2E_{,ij}\right] dx^i dx^j,$$

$\phi = \phi(t)$; B, ψ, E are all functions of t and x.

Choosing the gauge

$$\phi = 0 = E,$$

we have

III. Cosmology in Horava-Lifshitz theory (Cont.):

$$\ddot{\psi}_k + \omega_\psi^2 \psi_k = 0,$$
$$\omega_\psi^2 = \frac{\xi}{2-3\xi} k^2 \left(1 - \alpha_1 k^2 + \alpha_2 k^4\right).$$
$$B_k = -\frac{2-3\xi}{\xi k^2} \dot{\psi}_k, \ (\xi \neq 0),$$

$\alpha_1 \equiv (8g_2 + 3g_3)/(2\kappa^2); \alpha_2 \equiv (3g_8 - 8g_7)/(2\kappa^2)^2.$

Clearly, for $0 < \xi \leq 2/3$, the solution is stable in the IR. When $\alpha_2 > 0$ it is also stable in the UV. When $\alpha_2 \geq \alpha_1^2/4$ it is stable in all the energy scales.

III. Cosmology in Horava-Lifshitz theory (Cont.):

For $\xi = 0$, we have

$$\begin{aligned} \psi &= G(x), \\ \partial^2 B &= t\left(1 + \alpha_1\partial^2 + +\alpha_2\partial^4\right)\partial^2 G(x) \\ &\quad + H(x). \end{aligned}$$

Although $B(t,x)$ is a linearly increasing function, the gauge-invaraint quantities,

$$\Phi = \dot{B} = I(x), \quad \Psi = \psi = G(x),$$

are all non-growing. So, **it is stable for $\xi = 0$, too!!!**

III. Cosmology in Horava-Lifshitz theory (Cont.):

- **Ghost Problem:**

$$S_V \simeq -\int dt d^3x \left(\frac{2-3\xi}{\xi} \dot{\psi}^2 - \partial^2 \psi + ... \right)$$

which has a wrong sign for $0 \leq \xi \leq 2/3$.

- **A possible way out:** The Vainshtein mechanism.

III. Cosmology in Horava-Lifshitz theory (Cont.):

B. Cosmological Perturbations [AW & R. Maartens, arXiv:0907.1748]:

$$\begin{aligned} ds^2 \;&=\; a^2(\eta)\{-(1+2\phi)d\eta^2 + 2B_{|i}dx^i dt \\ &\quad + \left[(1-2\psi)\gamma_{ij} + 2E_{|ij}\right] dx^i dx^j\}, \end{aligned}$$

$\phi = \phi(\eta)$; B, ψ, E: all functions of η & x. "|": Covariant derivative to γ_{ij}.

— Choosing the gauge $\phi = 0 = E$, we worked out the general formulas for linear scalar perturbations, including the conservations laws.

III. Cosmology in Horava-Lifshitz theory (Cont.):

In particular, we found that higher derivatives produce anisotropy,

$$
\begin{aligned}
\Phi - \Psi &= -8\pi G a^2 \left(\Pi + \Pi_{HL}\right), \\
\Pi_{HL} &= -\frac{1}{8\pi G a^4}\left(\alpha_1 + \frac{\alpha_2}{a^2}\nabla^2\right)\nabla^2\psi,
\end{aligned}
$$

Π: anisotropic stress of matter; ∇^2: the Laplance operator of γ_{ij}; and the gauge invariant variables,

$$
\begin{aligned}
\Phi &= \phi + \mathcal{H}\left(B - E'\right) + \left(B - E'\right)', \\
\Psi &= \psi - \mathcal{H}\left(B - E'\right).
\end{aligned}
$$

III. Cosmology in Horava-Lifshitz theory (Cont.):

C. Scalar Field Perturbations [AW, R. Maartens & D. Wands, arXiv:0909.5167]:

The general action takes the form,

$$
\begin{aligned}
\mathcal{L}_M &= \frac{1}{2N^2}\left(\dot{\varphi} - N^i\nabla_i\varphi\right)^2 - V\left(g_{ij}, \mathcal{P}_n, \varphi\right), \\
V &= V_0(\varphi) + V_1(\varphi)\mathcal{P}_0 + V_2(\varphi)\mathcal{P}_1^2 \\
&\quad + V_3(\varphi)\mathcal{P}_1^3 + V_4(\varphi)\mathcal{P}_2 \\
&\quad + V_5(\varphi)\mathcal{P}_0\mathcal{P}_2 + V_6(\varphi)\mathcal{P}_1\mathcal{P}_2, \\
\mathcal{P}_0 &\equiv (\nabla\varphi)^2, \;\; \mathcal{P}_i \equiv \Box^i\varphi, \;\; \Box \equiv g^{ij}\nabla_i\nabla_j,
\end{aligned}
$$

$V_s(\varphi)$: arbitrary functions of φ only.

III. Cosmology in Horava-Lifshitz theory (Cont.):

- Anisotropy is produced purely by higher-order curvatures:

$$\Phi - \Psi = \frac{1}{a^2}\left(\alpha_1 \partial^2 + \frac{\alpha_2}{a^2}\partial^4\right)\psi. \tag{7}$$

- In the sub-horizon regime, the metric and scalar field modes oscillate independently:

$$\begin{aligned} u_k &\simeq \frac{u_0}{\sqrt{\omega_\varphi}} e^{i\omega_\varphi \eta}, \quad \chi_k \simeq \frac{\chi_0}{\sqrt{\omega_\psi}} e^{i\omega_\psi \eta}, \\ \omega_\varphi^2 &\simeq -\frac{2V_6 k^6}{a^4}, \quad \omega_\psi^2 \simeq \frac{\xi \alpha_2 k^6}{(2-3\xi)a^4}, \end{aligned} \tag{8}$$

$u_k \equiv a\delta\varphi_k; \chi_k \equiv a\psi_k.$

III. Cosmology in Horava-Lifshitz theory (Cont.):

- Scale-invariant primordial perturbations are produced even without inflation:

$$
\begin{aligned}
\omega_\varphi^2 &= 2k^2\left(-\frac{V_6}{a^4}k^4 + \frac{V_2 + V_4'}{a^2}k^2 + V_1\right) + a^2 V_0'' \\
&\simeq -\frac{2V_6 k^6}{a^4},
\end{aligned}
$$

$$
\Rightarrow \quad P_{\delta\varphi} = \frac{2\pi^2}{k^3}\mathcal{P}_{\delta\varphi}, \quad \mathcal{P}_{\delta\varphi}^{1/2} = \frac{M_{pl}}{2\pi}.
$$

[S. Mukohyama, arXiv:0904.2190] But, due to low-energy corrections, it is not exact.

III. Cosmology in Horava-Lifshitz theory (Cont.):

- Perturbations are adiabatic in large-scale during the slow-roll inflation:

$$\begin{aligned}
\delta p_{\varphi_{nad}} &\equiv \delta p_\varphi - \frac{\dot{\bar{p}}_\varphi}{\dot{\bar{\rho}}_\varphi}\delta\rho_\varphi = \delta p_{\varphi_{nad}}^{GR} + \delta p_{\varphi_{nad}}^{HL},\\
\delta p_{\varphi_{nad}}^{GR} &\equiv \frac{2}{3a^2}\left(2+\frac{\varphi''}{\mathcal{H}\varphi'}\right)\left[\varphi'\delta\varphi' - \left(\varphi'' - \mathcal{H}\varphi'\right)\delta\varphi\right],\\
\delta p_{\varphi_{nad}}^{HL} &\equiv \left(1+\frac{2\varphi''}{\mathcal{H}\varphi'}\right)\frac{V_4(\varphi)}{3a^4}\partial^4\delta\varphi,
\end{aligned}$$

$\delta p_{\varphi_{nad}}^{GR}$: the non-adiabatic pressure perturbations in GR; $\delta p_{\varphi_{nad}}^{HL}$: correction due to high-order curvature corrections.

III. Cosmology in Horava-Lifshitz theory (Cont.):

- Curvature Perturbation is constant in super-horizon region:

$$
\begin{aligned}
\zeta' &= -\frac{\mathcal{H}}{\bar{\rho}_\varphi + \bar{p}_\varphi}\delta p_{\varphi_{nad}} + \frac{1}{3}\partial^2\sigma_V + \frac{1}{3}Q^{HL}, \\
\sigma_V &\equiv \frac{1}{\varphi'}\delta\varphi + B, \\
Q^{\mathrm{HL}} &\equiv \frac{V_4}{a^2\varphi'^2}\partial^4\delta\varphi' + \frac{1}{\varphi'}\left[(2V_1 - 1)\partial^2 - \frac{2V_6}{a^4}\partial^6\right. \\
&\quad \left. -\frac{1}{a^2}\left(2V_2 + V_4' + \mathcal{H}\frac{V_4}{\varphi'}\right)\partial^4\right]\delta\varphi,
\end{aligned}
$$

IV. Conclusions & Future Work

Conclusions: The theory is promising, and offers several very attractive features, such as UV complete, scale-invaraint primordial perturbations can be produced even without inflation, constant curvature perturbations are possible at least for a single scalar field, and so on.

But, it also faces lot of challenges:

- Renormalizability beyond power-counting has not been worked out.

IV. Conclusions & Future Work (Cont.)

- The renormalization group (RG) flow of various coupling constants has not been investigated. In particular, recovery of GR in the IR relies on the assumption that the parameter ξ flows to zero in the IR.

- The limit of speed is subject to the RG flow, new ideas are needed in order to ensure that different species including those in the standard model of particle physics are somehow related to each other so that their limits of speed agree with the velocity of light within experimental limits.

IV. Conclusions & Future Work (Cont.)

- The ghost problem in the scalar sector. Vainshtein effect must be investigated for the scalar graviton in the limit $\xi \to 0$. Since the time kinetic term (together with gradient terms) of the scalar graviton vanishes in this limit, one has to take into account nonlinear interactions to see if it really decouples.
- Cosmological applications, such as BBN, CMB and large-scale structure formations.
- Solar system tests.
- ...

THANK YOU !!!