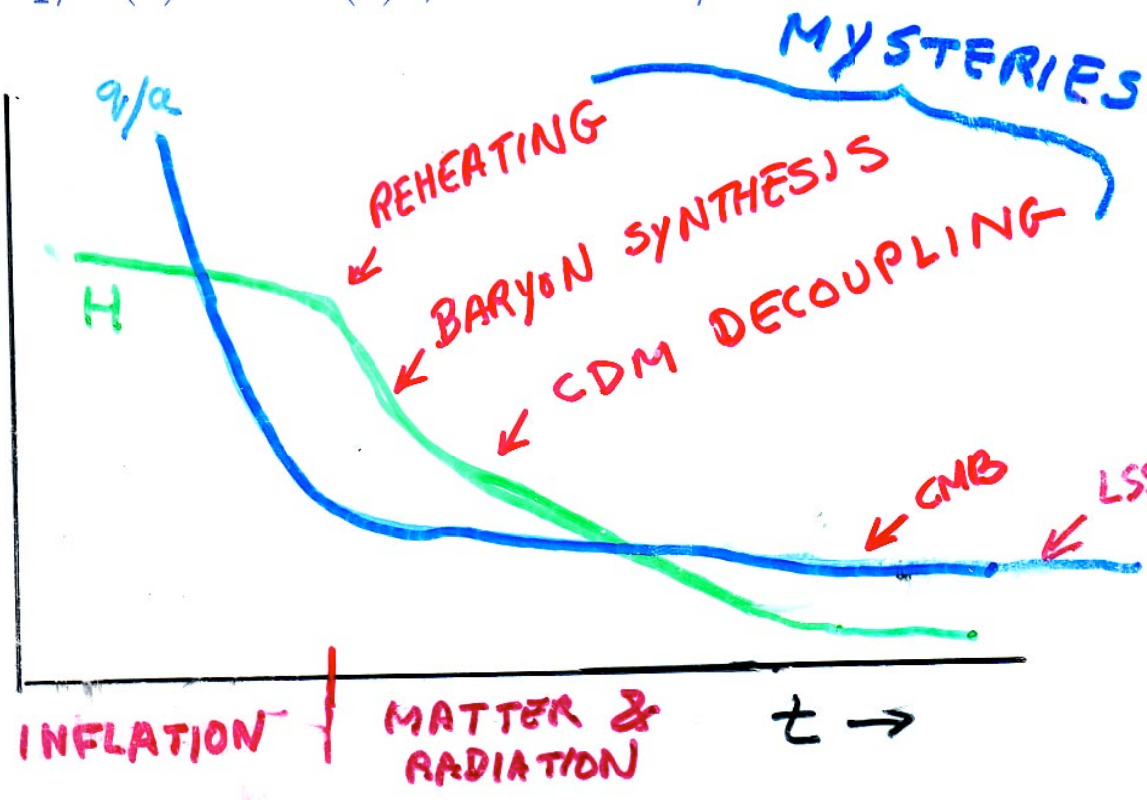
During most of the history of the universe, from well before the end of inflation until recently (i.e., $T \ll 10^{10}$K), the fluctuations we observe had physical wavelengths that were "outside the horizon," in the sense that the co-moving wave number q satisfied

$$q/a(t) \ll H(t) \ , \qquad H \equiv \dot{a}/a$$



During this era, there was one "adiabatic" scalar mode for which $\mathcal{R}$ was conserved: (J. Bardeen 1980)

$$\mathcal{R} \equiv -\Psi + H\delta u$$ (a.k.a. ζ)

(where 2Ψ = coefficient of δ_{ij} in δg_{ij}, and δu = velocity potential in $T^{\mu\nu}$). In adiabatic mode, $\delta\rho/(\bar{\rho}+\bar{p})$ is the same for all constituents of the universe, as seen in CMB. The universe stayed in this adiabatic mode if either

- Inflation was driven by the energy of a single "inflaton" field, or
- The universe ever spent a sufficiently long time in a state of thermal equilibrium with no non-zero values for conserved quantities.

Once adiabatic, it stays adiabatic!

Also, there is one tensor mode in which the gravitational wave amplitude γ is conserved outside the horizon (The only other mode decays rapidly.)

We understand how fluctuations evolve after they enter the horizon. So all we need to calculate microwave background anisotropies and large scale structure of matter (while still linear) are correlation functions:

$$\left\langle \mathcal{R}^o(\mathbf{x})\,\mathcal{R}^o(\mathbf{x}')\ldots\gamma^o(\mathbf{x}'')\ldots\right\rangle$$

at times (indicated by superscript "o") when wavelength is outside the horizon.

Here $\langle\ \ \rangle$ refers to expectation value in "in" vacuum, defined as state that appears like vacuum at early times, when $q/a \gg H$. So we want to calculate

$$\langle \text{vac, in}|\delta\phi_a(\mathbf{x},t)\,\delta\phi_b(\mathbf{y},t)\cdots|\text{vac, in}\rangle$$

not

$$\langle \text{vac, out}|T\Big\{\delta\phi_a(\mathbf{x},t)\,\delta\phi_b(\mathbf{y},t')\cdots\Big\}|\text{vac, in}\rangle\ ,$$

which is related to the S-matrix. And $|\text{vac, out}\rangle$ is not proportional to $|\text{vac, in}\rangle$!

($\delta\phi_a$ is any field: $\mathcal{R}$, γ^{ij}, etc.)

Schwinger "in-in" formalism:

$$\left\langle \text{vac, in} \middle| \delta\phi_a(\mathbf{x},t)\,\delta\phi_b(\mathbf{y},t)\cdots \middle| \text{vac, in} \right\rangle$$

$$= \left\langle \left[\bar{T}\exp\left(i\int_{-\infty}^{t} V^I(t')\,dt'\right)\right] \right.$$

$$\times\;\delta\phi_a^I(\mathbf{x},t)\,\delta\phi_b^I(\mathbf{y},t)\cdots$$

$$\left. \times \left[T\exp\left(-i\int_{-\infty}^{t} V^I(t')\,dt'\right)\right] \right\rangle_0$$

$\left\langle \ldots \right\rangle_0 \equiv$ bare vacuum expectation value.
$V \equiv$ terms in Hamiltonian of third or higher order in perturbations.
I indicates time-dependence generated by terms in Hamiltonian of second order in perturbations.

For comparison, in calculating the S-matrix we need

$$\left\langle \text{vac, out} \middle| \delta\phi_a(\mathbf{x},t)\,\delta\phi_b(\mathbf{y},t)\cdots \middle| \text{vac, in} \right\rangle$$

$$= \left\langle \delta\phi^I_a(\mathbf{x},t)\,\delta\phi^I_b(\mathbf{y},t)\cdots \right.$$

$$\left. \times\ T\exp\left(-i\int_{-\infty}^{t} V^I(t')\,dt'\right)\right\rangle_0$$

Graphical Interpretation:

Each vertex can be either of 2 kinds, L or R. With N vertices, we sum over 2^N possibilities, with an extra minus sign for each L vertex.

Propagators:

$\delta\phi_a^I(\mathbf{x}, t)$ L  L $\delta\phi_b^I(\mathbf{x}', t')$

$$\langle \bar{T}\{\delta\phi_a^I(\mathbf{x}, t)\, \delta\phi_b^I(\mathbf{x}', t')\}\rangle_0$$

$\delta\phi_a^I(\mathbf{x}, t)$ R  R $\delta\phi_b^I(\mathbf{x}', t')$

$$\langle T\{\delta\phi_a^I(\mathbf{x}, t)\, \delta\phi_b^I(\mathbf{x}', t')\}\rangle_0$$

$\delta\phi_a^I(\mathbf{x}, t)$ L  R $\delta\phi_b^I(\mathbf{x}', t')$

$$\langle \delta\phi_a^I(\mathbf{x}, t)\, \delta\phi_b^I(\mathbf{x}', t')\rangle_0$$

THEOREM: If $\delta\phi_a^{\rm cl}$ is the solution of the classical field equations with free field initial conditions for $q/a \gg H$, then the bare vacuum expectation value

$$\langle \delta\phi_a^{\rm cl}(\mathbf{x},t)\,\delta\phi_b^{\rm cl}(\mathbf{y},t)\cdots\rangle$$

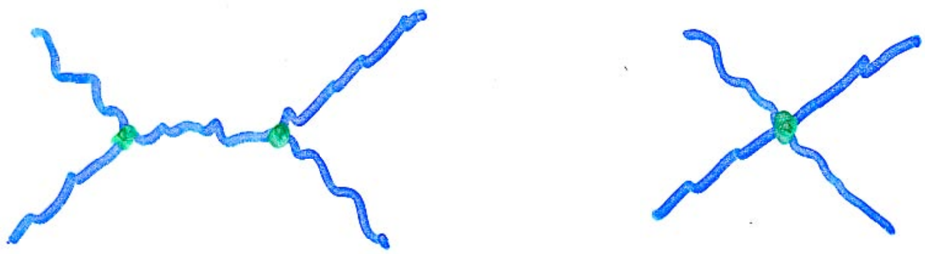
is the sum of the tree graphs for

$$\left\langle \text{vac, in}\middle|\,\delta\phi_a(\mathbf{x},t)\,\delta\phi_b(\mathbf{y},t)\cdots\middle|\text{vac, in}\right\rangle$$

[SW 2005] For instance,

Maldacena 2002 $\langle\mathcal{RRR}\rangle$ $\langle\mathcal{RR}\gamma\rangle$,&c [Also Gangui et al 1993; Creminelli & Zaldarriaga 2004; Rigopoulos et al 2004; Seery & Lidsey 2005]

TRISPECTRUM:



Correlation functions calculated in this way depend on the details of inflation only near the time of horizon exit. Non-Gaussian correlations are suppressed by factors of $GH_q^2 \approx 10^{-12}$, where H_q is Hubble constant at horizon exit. But this formalism is purely classical.

Cosmological non-classical fluctuations: Calzetta & Hu, 1987; Morikawa 1995; Tsamis & Woodard 1995, 1997, 1998; Onelmi & Woodard 2002, 2004; Prokopec, Tornkvist & Woodard, 2003; Prokopec & Woodard, 2003; Boyanovsky, Vega, & Sanchez 2005; Brunier, Onemli, & Woodard 2005; Collins & Holman 2005.

Cosmological Correlations

Do results depend on details of expansion only near horizon exit, like tree calculation, rather than over whole history of inflation? If so, L-loop corrections are suppressed by

$$(GH_q)^L \approx 10^{-12L} .$$

$H_q \equiv H(t)$ at horizon exit.

Example:Inflation with one inflaton, plus many massless scalar fields:

Co-moving gauge:

Inflaton: $\varphi(\mathbf{x},t)=\bar{\varphi}(t)$

ADM

$$g_{ij}=a^2(t)\delta_{ij}\Big(1+2\mathcal{R}(\mathbf{x},t)\Big)+\gamma_{ij}(\mathbf{x},t)+\dots$$

$$\partial_i\gamma_{ij}=\gamma_{ii}=0$$

$$g_{00}=-1-\frac{2\dot{\mathcal{R}}(\mathbf{x},t)}{H(t)}+\dots$$

$$g_{0i}=-\frac{\partial}{\partial x^i}\left(\frac{\mathcal{R}(\mathbf{x},t)}{H(t)}-a^2(t)\epsilon(t)\nabla^{-2}\dot{\mathcal{R}}(\mathbf{x},t)\right)+\dots$$

$$H\equiv\dot{a}/a\ ,\qquad \epsilon\equiv-\dot{H}/H^2\ .$$

$$\mathcal{L}_{\sigma\mathcal{R}\mathcal{R}} = -\frac{a}{2}\mathcal{R}\sum_n \partial_i\sigma_n\partial_i\sigma_n - \frac{a}{2H}\dot{\mathcal{R}}\sum_n \partial_i\sigma_n\partial_i\sigma_n$$

$$+\frac{a}{H}\partial_i\mathcal{R}\sum_n \dot{\sigma}_n\partial_i\sigma_n - \epsilon a^3\partial_i\left(\nabla^{-2}\dot{\mathcal{R}}\right)\sum_n \dot{\sigma}_n\partial_i\sigma_n$$

$$-\frac{a^3}{2H}\dot{\mathcal{R}}\sum_n \dot{\sigma}_n^2 + \frac{3a^3}{2}\mathcal{R}\sum_n \dot{\sigma}_n^2 \,.$$

$$\mathcal{R}^I(\mathbf{x},t) = \int d^3q\ \left[e^{i\mathbf{q}\cdot\mathbf{x}}\mathcal{R}_q(t)\alpha(\mathbf{q}) + \text{c.c.}\right]$$

$$\sigma_n^I(\mathbf{x},t) = \int d^3q\ \left[e^{i\mathbf{q}\cdot\mathbf{x}}\sigma_q(t)\alpha(n,\mathbf{q}) + \text{c.c.}\right]$$

For $q/aH \ll 1$,

$$\mathcal{R}_q \to \mathcal{R}_q^o + O\left(\frac{q^2}{a^2H^2}\right) \;, \quad \sigma_q \to \sigma_q^o + O\left(\frac{q^2}{a^2H^2}\right) \;.$$

$$\mathcal{L}_{\sigma\mathcal{R}\mathcal{R}} = -\frac{a}{2}\mathcal{R}\sum_n \partial_i\sigma_n\partial_i\sigma_n - \frac{a}{2H}\dot{\mathcal{R}}\sum_n \partial_i\sigma_n\partial_i\sigma_n$$

$$+\frac{a}{H}\partial_i\mathcal{R}\sum_n \dot{\sigma}_n\partial_i\sigma_n - \epsilon a^3\partial_i\left(\nabla^{-2}\dot{\mathcal{R}}\right)\sum_n \dot{\sigma}_n\partial_i\sigma_n$$

$$-\frac{a^3}{2H}\dot{\mathcal{R}}\sum_n \dot{\sigma}_n^2 + \frac{3a^3}{2}\mathcal{R}\sum_n \dot{\sigma}_n^2 \;.$$

To see cancellations, use the more convenient formula:

$$\left\langle \text{vac, in} \middle| \delta\phi_a(\mathbf{x},t)\,\delta\phi_b(\mathbf{y},t)\cdots \middle| \text{vac, in} \right\rangle$$

$$= \sum_{n=0}^{\infty} i^n \int_{-\infty}^{t} dt_n \int_{-\infty}^{t_n} dt_{n-1} \cdots \int_{-\infty}^{t_2} dt_1$$

$$\times \Big\langle \Big[V^I(t_1), \Big[V^I(t_2),$$

$$\cdots \Big[V^I(t_n), \delta\phi_a^I(\mathbf{x},t)\,\delta\phi_b^I(\mathbf{y},t)\cdots\Big]\cdots\Big]\Big]\Big\rangle_0$$

SW 2005

$$\Big[\mathcal{R}(\mathbf{x},t)\,,\,\mathcal{R}(\mathbf{y},t')\Big] = \int d^3q\; e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}$$

$$\Big(\mathcal{R}_q(t)\mathcal{R}_q^*(t') - \mathcal{R}_q(t')\mathcal{R}_q^*(t)\Big)$$

$$\mathcal{R}_q(t) \to \mathcal{R}_q^o\left[1 + \int_t^\infty \frac{q^2\,dt'}{a^3(t')\epsilon(t')}\right.$$

$$\left.\times \int_{-\infty}^{t'} a(t'')\,\epsilon(t'')\,dt'' + \ldots\right]$$

$$+\,\mathcal{C}_q\left[\int_t^\infty \frac{dt'}{a^3(t')\,\epsilon(t')} + \ldots\right]$$

$$[\mathcal{R},\mathcal{R}] \sim a^{-3}\,, \qquad [\mathcal{R},\dot{\mathcal{R}}] \sim a^{-3}$$

$$[\dot{\mathcal{R}},\dot{\mathcal{R}}] \sim a^{-5}$$

Likewise for σ_n.

Theorems:

1. In the theory of gravity plus scalars with purely gravitational interactions, the total number of factors of $a(t)$ in any subintegral over times is negative, so correlations receive no contribution from times long after horizon exit.

SW 2005

Example: $\mathcal{N} \gg 1$ scalars

$$\int d^3x \, e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \langle \text{vac, in} | \mathcal{R}(\mathbf{x},t)\, \mathcal{R}(\mathbf{y},t) | \text{vac, in} \rangle$$

$$= \frac{8\pi G H^2(t_q)}{4(2\pi)^3 |\epsilon(t_q)| q^3} \qquad \text{classical}$$

$$- \frac{\pi [8\pi G H^2(t_q)]^2 \mathcal{N}}{15(2\pi)^3 q^3} [\ln q + C] \qquad \text{one loop}$$

$$+ \dots ,$$

where $t_q \equiv$ time of horizon exit.

Result 1 does not apply with other interactions, such as:

$$\sqrt{\text{Det}\,g}\,V(\sigma)\ ,\ \sqrt{\text{Det}\,g}\,\sigma\bar{\psi}\psi\ ,\ \sqrt{\text{Det}\,g}\,A_\mu\bar{\psi}\gamma^\mu\psi$$

because

$$\sqrt{\text{Detg}} \propto a^3\ .$$

But instead we have:

2. In all theories, the total number of factors of $a(t)$ in any subintegral over times is at most zero, so the contribution to correlations from times long after horizon exit is at most enhanced by powers of $\ln a$. SW 2006