AST 376 Cosmology — Problem Set 3

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I. DARK ENERGY EQUATION OF STATE

Suppose dark energy has an equation of state $P_{\text{vac}} = w\rho_{\text{vac}}c^2$, where here w is supposed to be time dependent, i.e. $w = w(z)$. Show that the Hubble expansion rate, well after the radiation-dominated epoch, can be written as:

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_{\text{DE}} \exp\left[3\int_0^z [1+w(x)]d\ln(1+x)\right] ,$$

where Ω_{DE} is the fraction of critical density contributed by dark energy (DE) today. For simplicity, let's pick the following cosmological parameters: $\Omega_m = 0.3$, $\Omega_{\text{DE}} = 0.7$, $H_0 = 70$ km s^1 Mpc^1.

Answer: In order to derive the result we first consider conservation of energy in expanding space:

$$dE = -PdV \qquad \Rightarrow \qquad d\left(a^3\rho_i c^2\right) = -P_i da^3 \,. \tag{1}$$

For normal matter the e.o.s. is $P_m = 0$ so the solution is relatively simple:

$$d\left(a^3\rho_m c^2\right) = -P_m da^3 = 0 \qquad \Rightarrow \qquad \rho_m = \rho_{m,0}a^{-3} \,. \tag{2}$$

For dark energy with the time-dependent e.o.s. $P_{\text{vac}} = w(z)\rho_{\text{vac}}c^2$ the ODE becomes:

$$\begin{aligned}
& d\left(a^3\rho_{\text{vac}}c^2\right) = -P_{\text{vac}}da^3 = -w(a)\rho_{\text{vac}}c^2 da^3 \\
\Rightarrow \quad & a^{-3}\frac{d}{da}\left(a^3\rho_{\text{vac}}\right) = -w(a)\rho_{\text{vac}}a^{-3}\frac{d}{da}\left(a^3\right) \\
\Rightarrow \quad & \frac{d\rho_{\text{vac}}}{da} + 3\frac{\rho_{\text{vac}}}{a} = -\frac{3w(a)\rho_{\text{vac}}}{a} \\
\Rightarrow \quad & \frac{d\rho_{\text{vac}}}{\rho_{\text{vac}}} = -\frac{3[1+w(a)]da}{a} \\
\Rightarrow \quad & \rho_{\text{vac}} = \rho_{\text{vac},0}\exp\left[-3\int_1^a [1+w(a')]d\ln a'\right] \,.
\end{aligned} \tag{3}$$

We have used the chain rule to relate $d\ln a = da/a$.

Now the Friedmann equation is obtained by considering cosmic dynamics from Einstein's equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_{\text{eff}} = -\frac{4\pi G}{3}\left(\rho_m(a) + \rho_{\text{vac}}(a) + \frac{3P_{\text{vac}}(a)}{c^2}\right) = -\frac{4\pi G}{3}\Big(\rho_m(a) + \rho_{\text{vac}}(a)[1+3w(a)]\Big) \,. \tag{4}$$

We may simplify this equation by recalling

$$\Omega_m \equiv \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \qquad \text{and} \qquad \Omega_{\text{DE}} \equiv \frac{\rho_{\text{vac},0}}{\rho_{\text{crit},0}} \,, \qquad \text{where} \qquad \rho_{\text{crit},0} \equiv \frac{3H_0^2}{8\pi G} \,.$$

Thus, Eq. 4 becomes

$$\frac{\ddot{a}}{a} = -\frac{H_0^2}{2}\left(\Omega_m a^{-3} + \Omega_{\text{DE}}[1+3w(a)]\exp\left[-3\int_1^a [1+w(a')]d\ln a'\right]\right) \,. \tag{5}$$

Multiplying Eq. 5 by $2a\dot{a}H_0^{-2}$ gives

$$2\dot{a}\ddot{a}H_0^{-2} = -a\dot{a}\left(\Omega_m a^{-3} + \Omega_{\rm DE}[1+3w(a)]\exp\left[-3\int_1^a [1+w(a')]d\ln a'\right]\right) . \tag{6}$$

The term on the LHS of Eq. 6 can be integrated directly as follows:

$$\frac{1}{H_0^2}\int_{t_0}^t 2\dot{a}(t')\ddot{a}(t')dt' = \frac{1}{H_0^2}\int_{t_0}^t \frac{d}{dt'}\left[\dot{a}^2(t')\right]dt' = \left[\frac{\dot{a}^2(t')}{H_0^2}\right]_{t_0}^t = \frac{a^2H^2(t)}{H_0^2} - 1 . \tag{7}$$

Likewise for the first term on the RHS of Eq. 6:

$$-\Omega_m\int_{t_0}^t \frac{\dot{a}(t')dt'}{a^2(t')} = -\Omega_m\int_1^a \frac{da'}{a'^2} = -\Omega_m\left[-a'^{-1}\right]_1^a = \Omega_m a^{-1} - \Omega_m . \tag{8}$$

The last term on the RHS can be done using integration by parts. For simplicity we define $W(a) \equiv \exp[-3\int_1^a[1+w(a')]d\ln a']$ so that $W'(a) = -3a^{-1}[1+w(a)]W(a)$. From this the integration is $\int udv = uv - \int vdu$ where $u = a'^2$ and $v = W(a')$. The term without $\Omega_{\rm DE}$ becomes

$$\begin{aligned} -\int_1^a a'[1+3w(a')]W(a')da' &= \int_1^a 2a'W(a')da' + \int_1^a a'^2\frac{(-3)[1+w(a')]W(a')}{a'}da' \\ &= \int_1^a 2a'W(a')da' + \int_1^a a'^2\frac{dW(a')}{da'}da' \\ &= \int_1^a 2a'W(a')da' + \left[a'^2W(a')\right]_1^a - \int_1^a (2a')W(a')da' \\ &= \left[a'^2W(a')\right]_1^a = a^2W(a) - W(1) \\ &= a^2\exp\left[-3\int_1^a[1+w(a')]d\ln a'\right] - 1 . \end{aligned} \tag{9}$$

Finally, the Friedmann equation with a time-dependent e.o.s. is found by collecting these three terms:

$$\frac{a^2H^2(t)}{H_0^2} - 1 = \Omega_m a^{-1} - \Omega_m + a^2\Omega_{\rm DE}\exp\left[-3\int_1^a[1+w(a')]d\ln a'\right] - \Omega_{\rm DE} , \tag{10}$$

which is simplified by dividing by a^2 and rearranging terms:

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_m}{a^3} + \Omega_{\rm DE}\exp\left[-3\int_1^a[1+w(a')]d\ln a'\right] + \frac{(1-\Omega_m-\Omega_{\rm DE})}{a^2} . \tag{11}$$

The last term represents geometric curvature, but in a flat universe $\Omega_m + \Omega_{\rm DE} = 1$ so we have:

$$\boxed{\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_{\rm DE}\exp\left[3\int_0^z[1+w(x)]d\ln(1+x)\right] .} \tag{12}$$

Many experiments in the next decade, including UT's Hobby-Eberly Telescope Dark Energy Experiment (HETDEX), aim at constraining $w(z)$ to infer clues on the nature of dark energy. To understand how sensitive these measurements have to be, plot the:

(a) age of the universe:

(b) luminosity distance:

as a function of redshift for 4 different models: $w = 1$ (Einstein's cosmological constant), $w = -1/3$, $w = -0.5 + 0.1z$, $w = -0.5 - 0.05z$, for $0 < z < 5$.

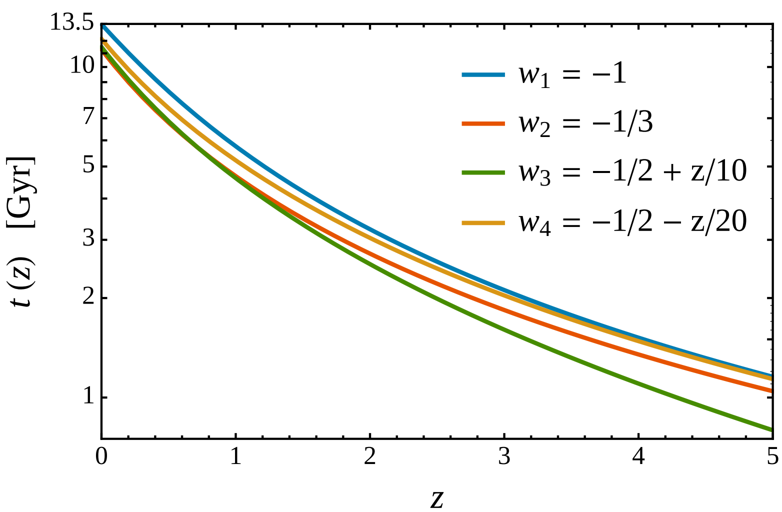
Answer: The age of the universe is given by solving the Friedmann equation, i.e. Eq. 12,

$$t(z) = \int_0^t dt' = \int_0^a \frac{da'}{a'H(a')} = \int_z^\infty \frac{dz'}{(1+z')H(z')} \qquad (\text{“Age of the Universe”})\,. \tag{13}$$

For the four models we have $W(z) \equiv \exp[3\int_0^z [1+w(x)]d\ln(1+x)]$:

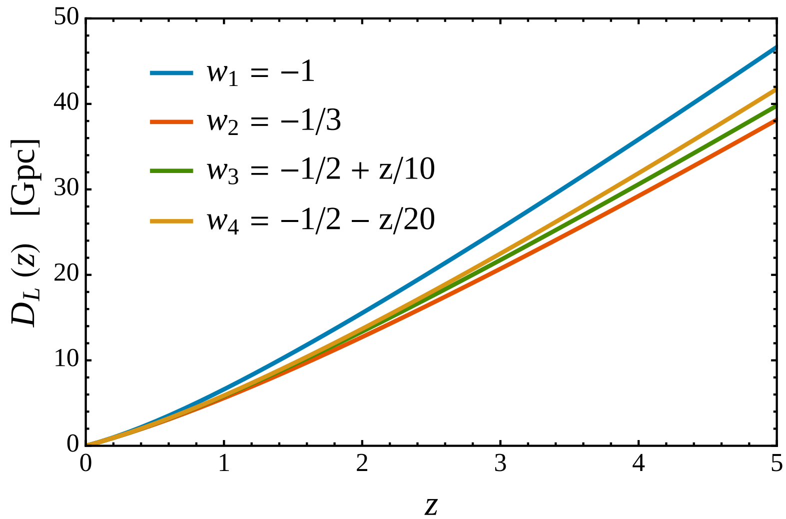
$$\begin{aligned}
w_1 &= -1 & W_1(z) &= 1 \\
w_2 &= -\frac{1}{3} & W_2(z) &= (1+z)^2 \\
w_3 &= -\frac{1}{2} + \frac{z}{10} & W_3(z) &= (1+z)^{6/5}\exp\left(\frac{3z}{10}\right) \\
w_4 &= -\frac{1}{2} - \frac{z}{20} & W_4(z) &= (1+z)^{33/20}\exp\left(-\frac{3z}{20}\right)
\end{aligned} \tag{14}$$

Using $H_i(z) = H_0[\Omega_m(1+z)^3 + \Omega_{\rm DE}W_i(z)]^{1/2}$ results in the following plot:



Likewise, the luminosity distance is calculated (and plotted) as follows:

$$D_{\rm L}(z) = (1+z)r(z) = (1+z)\int_0^z \frac{cdz'}{H(z')} \qquad (\text{“Luminosity distance”})\,. \tag{15}$$



II. FROM COSMIC DECELERATION TO ACCELERATION

Early on in cosmic history, dark energy was not important, and normal matter caused cosmic expansion to decelerate ($\ddot{a} < 0$). At some time, however, below a critical redshift, $z_{\rm crit}$, dark energy takes over, causing an accelerated expansion ($\ddot{a} > 0$).

(a) Using current cosmological parameters, calculate the value of $z_{\rm crit}$ (where $\ddot{a} = 0$).

Answer: Earlier in Eq. 5 we came across the following version of $\ddot{a}/a = -\frac{4\pi G}{3}\rho_{\rm eff}$:

$$\ddot{a} \propto \Omega_m(1+z)^3 + \Omega_{\rm DE}[1+3w(z)]\exp\left[3\int_0^z [1+w(x)]d\ln(1+x)\right] = 0\,. \tag{16}$$

This can be done analytically for the cosmological constant model where $w(z) = -1$:

$$0 = \Omega_m(1+z_{\rm crit})^3 - 2\Omega_{\rm DE} \qquad \Rightarrow \qquad \boxed{z_{\rm crit} = \left(\frac{2\Omega_{\rm DE}}{\Omega_m}\right)^{1/3} - 1 \approx 0.671\,.}$$

For the $w_2(z) = -1/3$ model there is no $z_{\rm crit}$ that works, but the w_3 and w_4 models have $z_{\rm crit} \approx 0.075$ and $z_{\rm crit} \approx 0.138$, respectively.

(b) How long ago was that (in units of Gyr)? I.e., calculate the *lookback time* to redshift $z_{\rm crit}$.

Answer: The lookback time is the difference in the ages of the universe at a redshift of z from the current age. In other words change the limits of integration to be from z to 0:

$$t_{\rm L}(z) = \int_t^{t_0} dt' = \int_a^1 \frac{da'}{a'H(a')} = \int_0^z \frac{dz'}{(1+z')H(z')} \qquad (\text{“Lookback time”})\,. \tag{17}$$

$$t_{\rm L}(z_{\rm crit}) = \int_0^z \frac{dz'}{(1+z')H_0\sqrt{\Omega_m(1+z)^3 + \Omega_{\rm DE}}} = \boxed{6.142\ \text{Gyr}}$$

There is no lookback time for the $w_2(z) = -1/3$ model but the corresponding results for the w_3 and w_4 models are 0.9784 Gyr and 1.692 Gyr, respectively.