AST 376 Cosmology — Problem Set 2

Prof. Volker Bromm — TA: Aaron Smith

I. COSMIC EXPANSION HISTORY

In class, we derived the Friedmann equation from the Einstein Field Equation.

(a) Show that the solution to the Friedmann equation can be written as:

$$t(z) = \frac{1}{H_0} \int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}} \,, \tag{1}$$

where $\Omega_m = 0.27$, $\Omega_\Lambda = 0.73$, $H_0 = 70$ km s^{-1} Mpc^{-1}, and $t(z)$ is the age of the universe.

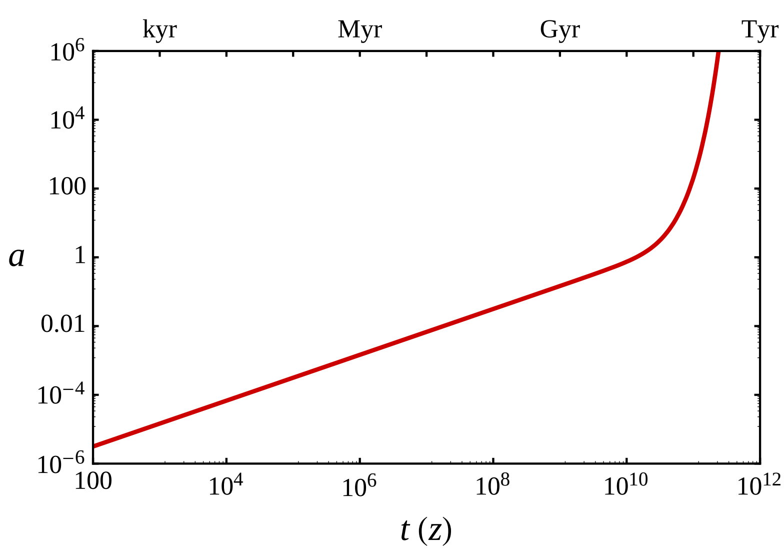
The Friedmann equation for a ΛCMD universe is

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_m a^{-3} + \Omega_\Lambda} \,.$$

We can substitute $a = (1+z)^{-1}$ and $\dot{a} dt = da = \frac{da}{dz} dz = -(1+z)^{-2} dz$ to arrive at

$$\begin{aligned} t(z) &= \int_0^{t(z)} dt \\ &= \int_0^{a(z)} \frac{da}{a} \frac{1}{H_0 \sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} \\ &= \int_\infty^z \frac{-(1+z')^{-2} dz'}{(1+z')^{-1}} \frac{1}{H_0 \sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}} \\ &= \frac{1}{H_0} \int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}} \,. \end{aligned}$$

(b) Now, evaluate Eq. 1 numerically, and plot the result with $a = (1+z)^{-1}$ on the y-axis and time t on the x-axis. Use a log-log scaling and choose the time units appropriately.



(c) For the matter dominated era ($1 \lesssim z \lesssim 1000$) Eq. 1 can be solved analytically. Derive an approximate solution, $t(z) \approx \ldots$ for this situation.

To proceed we evaluate the integral dropping the Ω_Λ term:

$$\begin{aligned} t(z) &= \frac{1}{H_0}\int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m(1+z')^3+\Omega_\Lambda}} \\ &\approx \frac{1}{H_0}\int_z^\infty \frac{dz'}{\sqrt{\Omega_m}(1+z')^{5/2}} \\ &\approx \frac{1}{H_0\sqrt{\Omega_m}}\left[\frac{-2}{3}(1+z')^{-3/2}\right]_z^\infty \\ &\Rightarrow \qquad \boxed{t(z) \approx \frac{2(1+z)^{-3/2}}{3H_0\sqrt{\Omega_m}}\,.} \end{aligned}$$

II. COSMOLOGICAL DISTANCES

The most distant gamma-ray burst (GRB) was recently detected at $z \approx 9.4$.

(a) How old was the universe at the time of the GRB (use units of Myr)?

The simple way to answer this is to use the approximation in part (c) above where it helps to know the Hubble time $H_0^{-1} = 1/(70 \text{ km s}^{-1} \text{ Mpc}^{-1}) = 13.978$ Gyr:

$$t(z=9.4) \approx \left.\frac{2(1+z)^{-3/2}}{3H_0\sqrt{\Omega_m}}\right|_{z=9.4} = \frac{2(13.987 \text{ Gyr})}{3\sqrt{0.27}(1+9.4)^{3/2}} = \boxed{534.7 \text{ Myr}}.$$

Note: The precise answer from the numerical integration is 534.5 Myr.

(b) What is the proper distance (in Gpc) to this GRB, evaluated at the present time, where $a_0 = 1$?

From the class notes we have:

$$r_0 = \int_0^z \frac{cdz'}{H(z')} = \int_0^z \frac{cdz'}{H_0\sqrt{\Omega_m(1+z')^3+\Omega_\Lambda}} = \boxed{9.668 \text{ Gpc}}.$$

The integral was evaluated numerically, e.g. in Mathematica this can be done with

```
Om = 0.27;
OL = 0.73;
c  = Quantity[3*^10, "Centimeters"/"Seconds"]
H0 = Quantity[70., "Kilometers"/"Seconds"/"Megaparsecs"]
rGpc[z_] := UnitConvert[c/H0 NIntegrate[1/Sqrt[Om (1+zp)^3+OL],{zp,0,z}],"Gigaparsecs"]
Print["The GRB at z=9.4 is currently ", rGpc[9.4], " away from us."]
```