AST 376 Cosmology — Problem Set 1

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I. NEWTONIAN COSMOLOGY

In class, we solved the Friedmann equation for the critical case, where the constant of integration was set to $k = 0$; this resulted in the Einstein-de Sitter model, where $a \propto t^{2/3}$. Now, let us consider the closed case ($k = +1$), where the universe starts with a Big Bang, reaches a maximum expansion, turns around, and eventually ends in a Big Crunch.

For the closed model, it is convenient to write the Friedmann equation as follows:

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\left(a^{-1} - 1\right) , \tag{1}$$

where ρ_0 is the present-day mass density.

(a) Parametric Expressions

The Friedmann equation can be solved with the following parametric expressions:

$$a = \sin^2\alpha \qquad \text{and} \qquad t = A(\alpha - \sin\alpha\cos\alpha) ,$$

where α is a "development angle", such that $\alpha = 0$ corresponds to the Big Bang, $\alpha = \frac{\pi}{2}$ corresponds to the point of maximum expansion ("turn-around"), and $\alpha = \pi$ to the Big Crunch.

To find the constant A we first find $\dot{a}$ from the parametric expressions. Consider that

$$\frac{da}{d\alpha} = \frac{d}{d\alpha}\left(\sin^2\alpha\right) = 2\sin\alpha\cos\alpha$$

and

$$\frac{dt}{d\alpha} = A\frac{d}{d\alpha}(\alpha - \sin\alpha\cos\alpha) = A\left(1 - \cos^2\alpha + \sin^2\alpha\right) = 2A\sin^2\alpha ,$$

so the derivative becomes

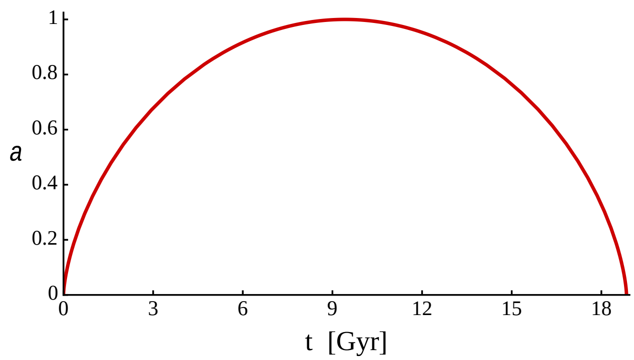
$$\dot{a} = \frac{da}{dt} = \frac{da/d\alpha}{dt/d\alpha} = \frac{2\sin\alpha\cos\alpha}{2A\sin^2\alpha} = \frac{1}{A\tan\alpha} .$$

Thus, according to Eq. 1 we have

$$\begin{aligned}
0 &= \left(\frac{1}{A\tan\alpha}\right)^2 - \frac{8\pi G\rho_0}{3}\left(\frac{1}{\sin^2\alpha} - 1\right) \\
&= \frac{\cot^2\alpha}{A^2} - \frac{8\pi G\rho_0}{3}\left(\csc^2\alpha - 1\right) \\
&= \cot^2\alpha\left(\frac{1}{A^2} - \frac{8\pi G\rho_0}{3}\right) \\
\Rightarrow &\qquad \boxed{A = \sqrt{\frac{3}{8\pi G\rho_0}}\,.}
\end{aligned}$$

(b) Plot

Plot this solution, showing the scale factor on the y-axis, and time on the x-axis. Assuming $\rho_0 = 5 \times 10^{-29}$ g cm^{-3}, show time in units of Gyr. (Recall that the scale factor is dimensionless.)



(c) Age of the Universe

What is the total duration of such a universe? I.e., what is the time elapsing between Big Bang and Big Crunch (in units of Gyr)?

We simply need to calculate the time when $\alpha = \pi$:

$$t_{\text{Big Crunch}} = \sqrt{\frac{3}{8\pi G\rho_0}}(\alpha - \sin\alpha\cos\alpha)\Big|_{\alpha=\pi} = \sqrt{\frac{3\pi}{8G\rho_0}} = 18.84 \text{ Gyr}\,. \tag{2}$$

(d) Einstein-de Sitter

Show that at early times, i.e., for small development angles α, the closed solution discussed here approximately approaches the Einstein-de Sitter scaling ($a \propto t^{2/3}$). To do this, Taylor expand (in α) the parametric expressions in part (a) above.

$$t \propto \alpha - \sin\alpha\cos\alpha \approx \alpha - \alpha\left(1 - \frac{\alpha^2}{2}\right) \propto \alpha^3 \qquad \text{and} \qquad a = \sin^2\alpha \approx \alpha^2$$

combine to give

$$a \propto t^{2/3}\,.$$

II. FLAT SPACETIME METRIC

The spatial (x, y, z) part of the Euclidean (flat) metric can be written, using spherical coordinates (r, θ, φ) as follows:

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2\,. \tag{3}$$

Explicitly show that this can be transformed into the standard Cartesian form, thus proving that the metric in Eq. 3 is flat.

To proceed we recall the definitions of (r, θ, φ) in terms of (x, y, z):

$$\begin{aligned} r &= \left(x^2 + y^2 + z^2\right)^{1/2} \\ \theta &= \cos^{-1} \frac{z}{\left(x^2 + y^2 + z^2\right)^{1/2}} \\ \varphi &= \tan^{-1} \frac{y}{x} \, . \end{aligned}$$

Therefore, using the chain rule (e.g. $dr = \frac{dr}{dx} dx + \frac{dr}{dy} dy + \frac{dr}{dz} dz$) the differentials are:

$$\begin{aligned} dr &= \frac{xdx + ydy + zdz}{r} \\ d\theta &= \frac{z\left(xdx + ydy\right) - \left(x^2 + y^2\right) dz}{r^2 \left(x^2 + y^2\right)^{1/2}} \\ d\varphi &= \frac{-ydx + xdy}{x^2 + y^2} \, . \end{aligned}$$

The form of the metric in Eq. 3 suggests what quantities we should calculate:

$$\begin{aligned} r^2 dr^2 &= \left(xdx + ydy + zdz\right)^2 \\ &= x^2 dx^2 + y^2 dy^2 + z^2 dz^2 + 2xy \; dxdy + 2xz \; dxdz + 2yz \; dydz \end{aligned}$$

$$\begin{aligned} r^4 d\theta^2 &= \frac{\left[z\left(xdx + ydy\right) - \left(x^2 + y^2\right) dz\right]^2}{x^2 + y^2} \\ &= \left(x^2 + y^2\right) dz^2 - 2xz \; dxdz - 2yz \; dydz + \frac{z^2 \left(x^2 dx^2 + 2xy \; dxdy + y^2 dy^2\right)}{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} r^4 \sin^2 \theta d\varphi &= \frac{r^2 \left(ydx - xdy\right)^2}{x^2 + y^2} \\ &= \left(1 + \frac{z^2}{x^2 + y^2}\right) \left(y^2 dx^2 - 2xy \; dxdy + x^2 dy^2\right) \\ &= y^2 dx^2 - 2xy \; dxdy + x^2 dy^2 + \frac{z^2 \left(y^2 dx^2 - 2xy \; dxdy + x^2 dy^2\right)}{x^2 + y^2} \, . \end{aligned}$$

Adding each of these together and collecting (by inspection) the dx^2, $dxdy$, etc. terms we have:

$$\begin{aligned} r^2 d\ell^2 &= r^2 dr^2 + r^4 d\theta^2 + r^4 \sin^2 \theta d\varphi \\ &= \left(x^2 + y^2\right)\left(dx^2 + dy^2 + dz^2\right) + z^2 dz^2 + \frac{z^2 \left(x^2 + y^2\right)\left(dx^2 + dy^2\right)}{x^2 + y^2} \\ &= r^2 \left(dx^2 + dy^2 + dz^2\right) \, . \end{aligned}$$

This proves the metric corresponds to flat space.