AST376 (Spring 2014)

COSMOLOGY

Problem Set 1

Due in class: Thursday, February 13, 2013

(worth 10/100)

1. Newtonian Cosmology

In class, we solved the Friedmann equation for the critical case, where the constant of integration was set to $k = 0$; this resulted in the Einstein-de Sitter model, where $a \propto t^{2/3}$. Now, let us consider the closed case ($k = +1$), where the universe starts with a Big Bang, reaches a maximum expansion, turns around, and eventually ends in a Big Crunch.

For the closed model, it is convenient to write the Friedmann equation as follows:

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\left(a^{-1} - 1\right) \ ,$$

where ρ_0 is the present-day mass density.

a. Show that this equation can be solved with the following parametric expressions:

$$a = \sin^2\alpha \ ,$$

and

$$t = A\left(\alpha - \sin\alpha\cos\alpha\right) \ ,$$

where α is a "development angle", such that $\alpha = 0$ corresponds to the Big Bang, $\alpha = \pi$ corresponds to the point of maximum expansion ("turn-around"), and $\alpha = 2\pi$ to the Big Crunch.

Find an expression for the constant A.

b. Plot this solution, showing the scale factor on the y-axis, and time on the x-axis. Assuming $\rho_0 = 5 \times 10^{-29}\,\mathrm{g\,cm^{-3}}$, show time in units of Gyr. (Recall that the scale factor is dimensionless.)

c. What is the total duration of such a universe? I.e., what is the time elapsing between Big Bang and Big Crunch (in units of Gyr)?

d. Show that at early times, i.e., for small development angles α, the closed solution discussed here appoximately approaches the Einstein-de Sitter scaling ($a \propto t^{2/3}$). To do this, Taylor expand (in α) the parametric expressions in part a above.

2. Flat Spacetime Metric

The spatial (x, y, z) part of the Euclidean (flat) metric can be written, using spherical coordinates $(r, \theta, \phi$a,) as follows:

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad . \tag{1}$$

Explicitly show that this can be transformed into the standard Cartesian form:

$$d\ell^2 = dx^2 + dy^2 + dz^2,$$

thus proving that the metric in equ. (1) corresponds to flat space.