AST 376 Cosmology — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith
(Dated: January 23, 2014)

NEWTONIAN COSMOLOGY

As discusses previously a Newtonian formulation of cosmology is not entirely consistent. However, if we work in the regime where $R_{\rm S} \ll R$ we can illustrate some correct aspects of the full theory (meaning GR). One of our main goals will be to find the evolution of the scale factor $a(t)$.

The cosmic equation of motion (e.o.m)

First of all we simplify things considerably by assuming $R_{\rm S}/R \ll 1$ so Newtonian physics is valid. We then consider a sphere of radius R and uniform density ρ. The mass within a shell is $\Delta M = 4\pi r^2 \rho \Delta r$ and the total mass is $M = \frac{4}{3}\pi R^3 \rho$. The e.o.m. is found by combining Newton's $2^{\rm nd}$ Law, $F = ma$, with the universal law of gravitation, $F \propto 1/r^2$,

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4}{3}\pi G \rho R \,. \tag{1}$$

However, this is in terms of the physical radius. We can get a relation for the scale factor by substituting $R = ax$ and $\ddot{R} = \ddot{a}x$ giving

$$\boxed{\ddot{a} = -\frac{4\pi G}{3}\rho a \,.} \tag{2}$$

Evolution of mass density

Consider a cubical region of the expanding space. Conservation of mass dictates that the mass ΔM within a given comoving volume is constant in time. Comparing the mass at two different times (t and t_0),

$$\Delta M(t) = a^3 x^3 \rho(t) \qquad \text{and} \qquad \Delta M(t_0) = a_0^3 x^3 \rho(t_0) = x^3 \rho_0 \,, \tag{3}$$

provides the evolution of the cosmic matter density

$$\boxed{\rho(t) = \rho_0 a^{-3} = \rho_0 (1+z)^3 \,.} \tag{4}$$

Note: This is only valid for normal (ordinary) nonrelativistic matter. We do not expect this to hold for radiation, dark energy, etc. Furthermore, the density is infinite at the Big Bang, i.e. in the limit as $z \to \infty$. From Eqs. 2 and 4 the full equation of motion is

$$\boxed{\ddot{a} = -\frac{4\pi G}{3}\rho_0 a^{-2} \qquad \text{(Matter equation of motion)} \,.} \tag{5}$$

Toward the simplified Friedmann equation

At this point we may use any method we like to solve the differential equation in Eq. 5. We proceed with an old trick to eliminate one of the time derivatives as follows:

(i) Multiply both sides by $2\dot{a}$ to get

$$2\dot{a}\ddot{a} = -\frac{8\pi G}{3}\rho_0 \dot{a} a^{-2} \,. \tag{6}$$

(ii) Use the fact that $\frac{d}{dt}\left(\dot{a}\right)^2 = 2\dot{a}\ddot{a}$ and $\frac{d}{dt}\left(a^{-1}\right) = -a^{-2}\dot{a}$ to rearrange Eq. 5 into

$$\frac{d\left(\dot{a}\right)^2}{dt} = \frac{8\pi G}{3}\rho_0 \frac{d\left(a^{-1}\right)}{dt} \,. \tag{7}$$

(iii) Integrate both sides, allow for a constant of integration k, and divide by a^2 to arrive at

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3} - \frac{k}{a^2} \qquad (\text{“simplified Friedmann Equation”})\,.} \tag{8}$$

Note: The constant of integration k represents the geometric curvature of the universe. The state of cosmology for most of the 20^{th} century was one where the value of k was quite uncertain. Fortunately, the theorist's perfect model of a flat ($k = 0$) universe turns out to correspond to ours.

Critical density

Assume the special case where the kinetic and potential energy budgets are exactly balanced:

$$\boxed{E_{\text{kin}} + E_{\text{pot}} = 0\,.} \tag{9}$$

This corresponds to a flat ($k = 0$) universe! If $E_{\text{kin}} < E_{\text{pot}}$ we end up with a “Big Crunch” and $E_{\text{kin}} \geq E_{\text{pot}}$ results in a “Big Freeze.” Consider again an expanding shell with radius R_0 and velocity $v_0 = H_0 R_0$, as determined by Hubble's law. Eq. 9 describes a case of critical density $\rho_{\text{crit},0}$:

$$\frac{1}{2}\Delta M v_0^2 = \frac{GM}{R_0}\Delta M \qquad \Rightarrow \qquad \frac{1}{2}\left(H_0 R_0\right)^2 = \frac{G}{R_0}\left(\frac{4\pi}{3}R_0^3 \rho_{\text{crit},0}\right)\,. \tag{10}$$

Thus, the Hubble constant and critical density can be defined in terms of each other:

$$\boxed{H_0^2 \equiv \frac{8\pi G}{3}\rho_{\text{crit},0}} \qquad \Rightarrow \qquad \boxed{\rho_{\text{crit},0} \equiv \frac{3H_0^2}{8\pi G} \sim 10^{-29} \text{ g cm}^{-3}\,.} \tag{11}$$

Note: We now have a more subtle picture of the universe with (cold) baryonic matter, dark matter, radiation, neutrinos, dark energy, and possibly some unknown species. We label each density with respective subscripts. For now we only need to remember $\boxed{\rho_m + \rho_\Lambda \sim \rho_{\text{crit},0}}$ and $\boxed{\rho_m \sim 0.3\rho_{\text{crit},0}}$. To be correct we would also need to write subscripts to denote present-day values, however, with so many zeros it is almost universally accepted to drop them all, except for the one on $\rho_{\text{crit},0}$.

With this in mind we define the present-day fractional contribution to the density, or the omega parameter Ω which we could again think of as having a zero subscript,

$$\boxed{\Omega \equiv \frac{\rho_0}{\rho_{\text{crit},0}} \quad (\text{“omega parameter”})} \qquad \Rightarrow \qquad \boxed{\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega a^{-3} - \frac{k}{a^2} \quad (\text{Friedmann Eq.})\,.} \tag{12}$$

The Einstein-de Sitter model of the Universe ($\Omega = 1$)

Q: how do we fix the constant of integration?
A: We need the scale factor to converge to a finite value, i.e. as $a \to \infty$ the value of $\dot{a} \to 0$. This is only possible if $k = 0$ which corresponds to a flat universe. This also means $\Omega = 1$ as can be discovered by plugging in the present-day values of a and $\dot{a}$ into Eq. 15. Therefore, the e.o.m. is

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-3}\,, \tag{13}$$

which is solved by plugging in a power-law ansatz:

$$a(t) = At^{\alpha} \qquad \Rightarrow \qquad \alpha^2 t^{-2} = \frac{H_0^2}{A^3} t^{-3\alpha} \qquad \Rightarrow \qquad \alpha = \frac{2}{3} \quad \text{and} \quad A = \left[\frac{3}{2}H_0\right]^{2/3}. \tag{14}$$

This leads to the solution for an Einstein-de Sitter (EdS) universe:

$$\boxed{a(t) = \left[\frac{3}{2}H_0\right]^{2/3} t^{2/3}} \qquad \text{or simply} \qquad \boxed{a(t) \propto t^{2/3} \qquad (\text{“Einstein-de Sitter universe”})\,.} \tag{15}$$

In this model the exact age of the universe is

$$t_{\mathrm{H,EdS}} = \frac{2}{3H_0} \sim 9\ \mathrm{Gyr}\,, \tag{16}$$

which is younger than the age of the oldest globular clusters. Hence, we cannot live in an Einstein-de Sitter universe without suffering from a “cosmic age crisis.” Cosmic acceleration from dark energy, of course, fixes this problem.