AST 376 Cosmology — Lecture Notes

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COSMIC DYNAMICS (CONTD.)

Overview of Exam 1

When: Thursday, March 6$^{\text{th}}$
Softball Qs: Know the RW metric! & estimates for the age (H_0^{-1}) and size (c/H_0) of the Universe.
Other Qs: Similar to *(i)* Problem Set 1 *(ii)* Problem Set 2 *(iii)* Quiz 5 and *(iv)* mystery question.

Review

The idea of the previous lectures was to introduce the various contributions to the cosmic energy budget. We will discuss some components in greater detail later.

Always remember that redshift z and scale factor a are different parameters that describe the same thing, i.e. $\boxed{a = (1+z)^{-1}}$. Because of this we can say $\boxed{H(z) = H(a) = H(t)}$, where we originally introduced the Hubble parameter by way of Hubble's law: $\boxed{v = H(t)r}$. Essentially, if the physical coordinate r is related to the comoving or present day coordinate r_0 by $\boxed{r = ar_0}$ then the velocity is $v = \dot{a}r_0 = H(t)ar_0$ and $\boxed{\dot{a}/a}$. But the evolution of the Hubble parameter (a kind of rate of expansion) is given by the Friedmann equation: $\boxed{H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda}}$.

The radiation, matter, and dark energy dominated models

(a) Radiation-dominated The epoch when matter and radiation contribute equally to the cosmic energy budget can be determined by equating the redshift-dependent densities

$$\rho_r(t) \sim \rho_m(t)$$
$$\rho_{r,0}(1+z)^4 \sim \rho_{m,0}(1+z)^3\,,$$

but $z \gg 1$ so this gives a value of

$$\boxed{z_{\text{eq}} \sim \frac{\Omega_m}{\Omega_r} \sim 3400 \qquad \text{(“epoch of matter-radiation equality”)}\,.} \tag{1}$$

Thus, for $z > z_{\text{eq}}$ we can approximate the Friedmann equation by

$$\frac{\dot{a}}{a} \approx H_0\sqrt{\Omega_r}(1+z)^2 = H_0\sqrt{\Omega_r}a^{-2}\,,$$

so by separation of variables $\int a da \propto \int dt$. The scale factor in the radiation regime is given by

$$\boxed{a(t) \propto t^{1/2} \qquad \text{(“Radiation-dominated”)}\,.} \tag{2}$$

Note: In this model the constant of proportionality is $\sqrt{2H_0\sqrt{\Omega_r}}$.

(b) Matter-dominated After the epoch of matter-radiation equality $z_{\rm eq}$ the Universe evolves in a condition where $\rho_m \gtrsim \rho_r, \rho_{\rm de}$, which lasts until the matter density falls below the density of dark matter, so if $\Omega_m \equiv \rho_{m,0}/\rho_{\rm crit,0}$ and $\Omega_\Lambda \equiv \rho_{\rm de,0}/\rho_{\rm crit,0}$ then

$$\rho_m(t) \sim \rho_{\rm de}(t) \qquad \Omega_m(1+z)^3 \sim \Omega_\Lambda \,,$$

and the transition redshift is

$$\boxed{z_{\rm m} \sim \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} - 1 \sim 0.3 \qquad (\text{“epoch of matter-dark energy equality”})\,.} \tag{3}$$

Thus, for $z_{\rm m} < z < z_{\rm eq}$ we can approximate the Friedmann equation by

$$\frac{\dot{a}}{a} \approx H_0\sqrt{\Omega_m}(1+z)^{3/2} = H_0\sqrt{\Omega_m}a^{-3/2}\,,$$

so by separation of variables $\int \sqrt{a}da \propto \int dt$. The scale factor in the matter regime is given by

$$\boxed{a(t) \propto t^{2/3} \qquad (\text{“Matter-dominated”})\,.} \tag{4}$$

Note: In a matter only model the constant of proportionality is $\left(\frac{3}{2}H_0\sqrt{\Omega_m}\right)^{2/3}$.

(c) Dark energy-dominated We are already in an era when dark energy dominates the cosmic energy budget, which turns out to have drastic consequences about the end state of our Universe.

$$\frac{\dot{a}}{a} \approx H_0\sqrt{\Omega_\Lambda}\,,$$

may be solved by separation of variables $\int \frac{da}{a} = H_0\sqrt{\Omega_\Lambda}\int dt \Rightarrow \log a = H_0\sqrt{\Omega_\Lambda}(t-t_0)$ to give

$$\boxed{a(t) \propto \exp\left[H_0\sqrt{\Omega_\Lambda}\,t\right] \qquad (\text{“de Sitter Universe”})\,.} \tag{5}$$

Note: In a dark-energy only model the constant of proportionality is $\exp\left(-H_0\sqrt{\Omega_\Lambda}t_0\right)$.

This exponential growth is reminiscent of hyperinflation in economics and also represents the “inflationary phase” of the early universe. However, inflation and dark energy are **not** the same! **Note:** The Hubble parameter in the de Sitter model is constant: $H(z) = \dot{a}/a = H_0\sqrt{\Omega_\Lambda} =$ constant. Usually we expect the energy density to slow down expansion, but in this case we have an exotic cosmic acceleration! At some point the super luminous expansion will outpace what we can see. Our choices become more limited and eventually we will only be able to observe stars in our locally bound system. We live at an opportune time in the development of observational cosmology. It is a so called *cosmic coincidence* that we happen to live at the point where $\Omega_\Lambda \sim \Omega_m$ are the same order of magnitude.

Age of the Universe

We want a complete understanding of the history of the universe, which for now means of $a(t)$. For simplicity, we use the economical Friedmann equation:

$$H(t) = \frac{\dot{a}}{a} = H_0\sqrt{\Omega(z)} \qquad (\text{“Economical Friedmann equation”})\,. \tag{6}$$

By separation of variables, $\int_0^a da'/a'\sqrt{\Omega(z')} = H_0 \int_0^t dt'$, which we may solve numerically. Recasting $a = (1+z)^{-1}$ and $da = -(1+z)^{-2}dz$ gives the age of the Universe as a function of redshift z:

$$\boxed{t(z) = \frac{1}{H_0}\int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega(z')}} \qquad (\text{“Age of the Universe”})\,.} \tag{7}$$

The present age of the Universe is $t_0 = t(z=0) \approx 13.8$ Gyr. This is an incredible accomplishment. If there is one question in science it should be: “How old is the Universe?” Instead of a 10−20 Gyr uncertainty we know the value to $t_0 = 13.798 \pm 0.037$ Gyr.

Proper distance

Recall the example of the QSO and the main steps to relate proper distance with redshift:

(i) Get rid of the angular dependence in the RW metric: $ds^2 = -c^2dt^2 + a^2dr^2$.

(ii) Photons travel on null geodesics: $ds^2 = 0 \quad \Rightarrow \quad cdt = adr$.

If we had a simple power law (e.g. Einstein-de Sitter) we would already know $a(t)$ but instead we know $t(z)$ so we have to rework the proper distance as

$$r(z) = \int_{t(z)}^{t_0} \frac{cdt}{a(t)} = c\int_z^0 (1+z')\,\frac{dt}{dz'}dz' \quad \text{where} \quad \frac{dt}{dz} = \frac{d}{dz}\int_z^\infty \frac{dz'/H_0}{(1+z')\sqrt{\Omega(z')}} = \frac{-1/H_0}{(1+z)\sqrt{\Omega(z)}}\,.$$

Therefore the proper, or comoving, distance is

$$\boxed{r(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega(z')}} \qquad (\text{“Proper distance”})\,.} \tag{8}$$

There is one law of physics. Don't do the calculation before you know the answer. For example, we know that high-redshift QSOs ($z \approx 7$) are near the “edge of the Universe” so $r_{\rm QSO} \sim 9$ Gpc.