AST 376 Cosmology — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith
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COSMIC DYNAMICS (CONTD.)

Review

Modern cosmology assumes a Friedmann-Lemaitre-Robertson-Walker (FLRT) cosmology, which is based on the RW metric and the evolution of $a(t)$ from the Friedmann equation. Recall that the Friedmann equation has the exact same form as the Newtonian version:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad (\text{“Friedmann equation”})\,.} \tag{1}$$

So why did we need GR if the Newtonian arguments gave us the exact same equation? First of all, k is not a prediction of Newtonian theory, it is a free parameter that we have to fix. In this regard Einstein's theory has the powerful property that it gives physical meaning for k, which is the curvature of the Universe. We use a flat spatial geometry so $k = 0$. Furthermore, GR gives additional sources of gravity so the effective density also has a pressure term:

$$\rho_{\text{eff}} \equiv \rho + \frac{3P}{c^2} \qquad (\text{“effective density”})\,. \tag{2}$$

Finally, GR allows for density contributions from radiation ρ_r and vacuum energy ρ_{de}.

The density components

Vacuum energy: This is the energy attached to empty space. In cosmology there are many models for dark energy (e.g. Einstein's cosmological constant Λ, quintessence based on scalar fields, a time-dependent equation of state, etc.). The standard model, however, maintains a constant density

$$\boxed{\rho_{\text{de}}(t) = \rho_{\text{de},0} = \text{constant}\,.} \tag{3}$$

Recall the behavior of normal matter under expansion. Usually “adiabatic cooling” means the material experiences a loss of internal (thermal) energy according to $P\Delta V = -\Delta E$. Dark energy, on the other hand, gains energy upon expansion! Consider that the change in energy is

$$\Delta E = \rho_{\text{de},0} c^2 \Delta V = -P_{\text{de}} \Delta V\,.$$

In order to not violate energy conservation we need negative pressure(!)

$$\boxed{P_{\text{de}} = -\rho_{\text{de}} c^2 \qquad (\text{“negative pressure”})\,.} \tag{4}$$

No self-respecting substance we know of does this! The “tension” (negative pressure) of dark energy is apparently exactly satisfied by the work from expansion.

Matter ("cold" matter): By cold we mean the particles are non relativistic, i.e. $v \ll c$. There are two components: *(i)* cold dark matter (CDM) and *(ii)* "normal" baryonic matter, i.e. protons and neutrons. The density evolution as a function of scale factor and redshift are given by

$$\boxed{\rho_m = \rho_{m,0} a^{-3} = \rho_{m,0}(1+z)^3 \,.} \tag{5}$$

Radiation ("hot" matter): By hot we mean relativistic particles, i.e. v is (smaller but) on the order of the speed of light. This refers to photons, neutrinos, and extra radiation species. However, in practice this is the cosmic microwave background (CMB) which is (loosely) the remnant "thermal radiation" from the Big Bang. The CMB is the best blackbody we know about and has a characteristic temperature today of ~ 2.7 K. The evolution of radiation density is

$$\boxed{\rho_r = \rho_{r,0} a^{-4} = \rho_{r,0}(1+z)^4 \,.} \tag{6}$$

Radiation in thermal equilibrium follows the Stefan-Boltzmann law

$$u = \frac{\Delta E}{\Delta V} = a_{\rm rad} T^4 \qquad (\text{“Stephan-Boltzmann law”}) \tag{7}$$

where $a_{\rm rad} \equiv 4\sigma/c = 7.56 \times 10^{-15}$ erg cm^{-3} K^{-4} is the radiation constant. Therefore, the temperature of the CMB is inversely proportional to the scale factor because the energy density is

$$\boxed{u = c^2 \rho_r} \qquad \Rightarrow \qquad \boxed{T \propto a^{-1} \propto (1+z) \qquad (\text{“CMB temperature scaling”}) \,.} \tag{8}$$

The economical Friedmann equation

We now define the densities in terms of the critical density:

$$\boxed{\Omega_m \equiv \frac{\rho_{m,0}}{\rho_{\rm crit,0}} \qquad \Omega_r \equiv \frac{\rho_{r,0}}{\rho_{\rm crit,0}} \qquad \Omega_\Lambda \equiv \frac{\rho_{\rm de,0}}{\rho_{\rm crit,0}} \qquad \text{where} \qquad \rho_{\rm crit,0} \equiv \frac{3H_0^2}{3\pi G} \,.} \tag{9}$$

Note: The convention is to drop the '0' subscripts even though the Ω_i represent present day values. Therefore, the Friedmann equation is

$$\boxed{\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_r (1+z)^4 + \Omega_\Lambda} \qquad (\text{“Friedmann equation”}) \,.} \tag{10}$$

Finally, if we define $\Omega(z) \equiv \Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda$ and recall that $H(t) \equiv \frac{\dot{a}}{a}$ then we have

$$\boxed{\frac{H(t)}{H_0} = \sqrt{\Omega(z)} \qquad (\text{“Economical Friedmann equation”}) \,.} \tag{11}$$

Note: Plugging in present day values gives $\Omega(z=0) = \Omega_m + \Omega_r + \Omega_\Lambda = 1$ for a flat Universe.

Cosmological parameters

We are in an era of precision cosmology, which is possible by using combined Supernova (SN; e.g. with HST) and CMB (e.g. with WMAP, Planck, high-ℓ) data. The precision values are $\Omega_m \approx 0.307$, $\Omega_\Lambda \approx 0.692 \pm 0.010$, $\Omega_r \approx 9 \times 10^-5 \approx 0$, and $H_0 \approx 67.8 \pm 0.77$ km s^{-1} Mpc^{-1}. It is usually good enough to ignore radiation and assume

$$\boxed{\Omega_\Lambda \sim 0.7 \,, \qquad \Omega_m \sim 0.3 \,, \qquad \text{and} \qquad H_0 \sim 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \qquad (\text{“}\Lambda\text{CDM model”}) \,.} \tag{12}$$

Solutions to the Friedmann equation

The redshift dependence of ρ_r, ρ_m, and $\rho_{\rm de}$ naturally leads to three regions: (a) radiation-dominated, (b) matter-dominated, and (c) dark energy-dominated. The exact solution is best understood when considering these isolated situations.

(a) Radiation-dominated The epoch when matter and radiation contribute equally to the cosmic energy budget can be determined by equating the redshift-dependent densities

$$\begin{aligned} \rho_r(t) &\sim \rho_m(t) \\ \rho_{r,0}(1+z)^4 &\sim \rho_{m,0}(1+z)^3 \,, \end{aligned} \tag{13}$$

but $z \gg 1$ so this gives a value of

$$\boxed{z_{\rm eq} \sim \frac{\Omega_m}{\Omega_r} \sim 3400 \qquad (\text{“epoch of matter-radiation equality”})\,.} \tag{14}$$

Thus, for $z > z_{\rm eq}$ we can approximate the Friedmann equation by

$$\frac{\dot{a}}{a} \approx H_0\sqrt{\Omega_r}(1+z)^2 = H_0\sqrt{\Omega_r}a^{-2}\,,$$

so by separation of variables $\int a da \propto \int dt$. The scale factor in the radiation regime is given by

$$\boxed{a(t) \propto t^{1/2} \qquad (\text{“Radiation-dominated”})\,.} \tag{15}$$

Note: In this model the constant of proportionality is $\sqrt{2H_0\sqrt{\Omega_r}}$.