AST 376 Cosmology — Lecture Notes

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COSMIC SPACETIME (CONTD.)

Review

Last time we unveiled the metric of the Universe:

$$\boxed{ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + a^2\left(dr^2 + r^2d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\right) \qquad (\text{"Robertson-Walker"})\,.} \tag{1}$$

We have gone from ten unknowns in the metric $g_{\mu\nu}$ to just one time-dependent scale factor $a(t)$!

Q: Does the Robertson-Walker (RW) metric provide any new information about the Universe?
A: Yes! The metric gives us access to proper (or physical) distances. To motivate this we consider the object ULAS J1120+0641, which happens to be the most distant known quasar in the Universe. The spectra from the QSO (quasi-stellar object) reveals the flux – or what is measured through the detectors in our telescopes – as a function of wavelength. The figure shown in class shows the distinct Lyman-alpha (Lyα) line at the observed wavelength of $\lambda_{\text{obs}} \approx 1\ \mu$m, however, the Ly$\alpha$ rest wavelength was emitted at $\lambda_{\text{em}} = 1216$ Å $= 0.1216\ \mu$m. This implies a redshift of $z = \lambda_{\text{obs}}/\lambda_{\text{em}} - 1 \approx 7.1$ due to Hubble expansion.

Note: The unit of Angstrom (1 Å $\equiv 10^{-10}$ m) corresponds to the typical size of atoms.

There are many distances in cosmology (e.g. proper and comoving distances, light propagation distances, characteristic scales, sound horizons, the angular diameter distance, etc.)! We want to relate these distances to each other in order to understand their context. Fortunately, we already know how the proper distance changes with time. If the present day distance is r_0 then the proper distance $r(t)$ at any other time is proportional to the scale factor, i.e. $r(t) = r_0a(t)$.

Recall: The present day values for time, scale factor, and redshift are $t = t_0$, $a = a(t_0) = a_0 \equiv 1$, and $z = z(t_0) = z_0 = 0$, respectively. Also, a and z are related by $a = (1 + z)^{-1}$ and $z = a^{-1} - 1$.

Measuring distances on cosmological scales is not a straightforward task! We may locally ignore the effects of spacetime expansion and in some cases obtain high-precision measurements directly. However, there is no 'measuring stick' that instantaneously allows us to read off the distance to ULAS J1120+0641! For such distant objects we must appeal to calibrated methods inferred by the 'distance ladder' which takes advantage of so-called 'standard candles'. In our case we only have the redshift of the QSO, but this is enough if we also have a good model for the evolution of the Universe, i.e. an expression for $a(t)$. Without this realistic model all we know is that the light propagation distance cdt is shorter than the current proper distance r_0 and longer than proper distance $r(z = 7.1)$ when the original light was emitted. This is because the Universe has undergone expansion over the course of the several Gyr journey the photons took to reach our detectors. In general, although we still need to know $a(t)$ the distances are related by

$$\boxed{r_0 > cdt > r(z) = \frac{r_0}{1+z}\,.} \tag{2}$$

The proper length

As an example of how to use the RW metric we now consider the proper length ℓ to the QSO. This corresponds to a constant time, or $dt = 0$ in Eq. 1. By orienting the z-axis along the direction of the light from the QSO to us gives $d\vartheta = d\varphi = 0$. Together this means proper lengths follow

$$d\ell^2 = ds^2 = a^2 dr^2 \,,$$

which can be integrated from the QSO ($r = 0$) to us ($r = r_0$) to give

$$\boxed{\ell(t) = \int_0^{r_0} a(t) dr = r_0 a(t) \qquad (\text{“proper length”})\,.} \tag{3}$$

Therefore, proper lengths are described by physical coordinates and at present $\ell_0 = r_0$.

The light propagation distance

We only know about the QSO spectrum because of the light we observe. These photons travel along null geodesics so we may set the line element to zero,

$$ds^2 = -c^2 dt^2 + a^2(t) dr^2 = 0 \qquad \Rightarrow \qquad c dt = a(t) dr \,,$$

and use separation of variables to find the integrated distance from the QSO to us:

$$\boxed{r_0 = \int_0^{r_0} dr = \int_{t(a)}^{t_0} \frac{c dt'}{a(t')} = \int_a^1 \frac{c da'}{a'^2 H(a')} = \int_0^z \frac{c dz'}{H(z')} \qquad (\text{“comoving distance”})\,.} \tag{4}$$

In the last equalities we have used that $\dot{a} = da/dt = aH(t)$ and $da/dz = -a^2$. To compute the integral we need to solve for $a(t)$ from the Einstein equations, which is done in the next section.

COSMIC DYNAMICS

In order to solve the Einstein field equations we need to use the machinery of differential geometry that we discussed briefly before. We do not have time to cover this in full detail. Instead, we point to Appendix C of Oyvind Gron/Arne Naess: “Einstein’s Theory” (Springer) available through the UT Austin online library system. There the RW metric is used to calculate Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ (combinations of first derivatives $\frac{\partial g_{\mu\nu}}{\partial x^\alpha}$) and then the Ricci tensor $R_{\mu\nu}$ (combinations of first and second derivatives). Finally, this geometric LHS is set equal to the physics from the stress energy tensor $T_{\mu\nu}$ on the RHS. After some tedious algebra we find two independent equations:

$$2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} = 4\pi G \left(\rho - \frac{P}{c^2} \right) \tag{5a}$$

$$3\frac{\ddot{a}}{a} = -4\pi G \left(\rho + \frac{3P}{c^2} \right) \tag{5b}$$

These are just intermediate equations but there is a striking connection to the Newtonian expression, i.e. recall from before that $\ddot{a} = \frac{-GM}{a^2} = \frac{-4\pi G}{3}\rho a$. Thus, the crucial addition in Einstein’s theory is to replace density with an “effective density” defined by

$$\rho_{\text{eff}} \equiv \rho + \frac{3P}{c^2} \qquad (\text{“effective density”})\,. \tag{6}$$

We can simultaneously eliminate the pressure and $\ddot{a}$ terms to combine this into one equation:

$$\boxed{\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad (\text{"Friedmann equation"})\,.}} \tag{7}$$

Sometimes this is called the Friedmann-Lemaitre (FL) equation and the various models based on versions of this equation are "FL models". Finally, modern cosmology is roughly synonymous with "FLRW cosmology" where RW refers to the metric and FL describes solutions of $a(t)$.

Q: What is the density ρ in the Friedmann equation?
A: The main contributions come from matter ρ_m (cold dark matter and baryons), radiation ρ_r (CMB photons), and dark energy $\rho_{\rm de}$ (or vacuum energy), yeilding

$$\rho = \rho(t) = \rho_m + \rho_r + \rho_{\rm de}\,. \tag{8}$$

How do these terms evolve with time?

Matter (DM + baryons): Consider a box expanding in time. Mass conservation tells us the number of particles in a comoving volume is constant in time, but Mass = Density × Volume so

$$M_0 = M(t) \qquad \Rightarrow \qquad \rho_{m,0}a_0^3 = \rho_m a^3$$

gives the evolution of matter density as a function of scale factor or redshift:

$$\boxed{\rho_m = \rho_{m,0}a^{-3} = \rho_{m,0}(1+z)^3\,.} \tag{9}$$

Radiation (CMB photons): Although photons are also conserved their mean energy is reduced with redshift according to $\langle\epsilon_\gamma\rangle = h\nu = \frac{hc}{\lambda} \propto a^{-1}$. Therefore, because the energy density of radiation is $\rho_r c^2 = n_\gamma\langle\epsilon_\gamma\rangle$ the evolution of radiation density is

$$\boxed{\rho_r = \rho_{r,0}a^{-4} = \rho_{r,0}(1+z)^4\,.} \tag{10}$$

Vacuum energy (dark energy): Finally, because the "energy of space itself" is only concerned about how much space is physically available, the vacuum energy density is constant in time

$$\boxed{\rho_{\rm de} = \rho_{\rm de,0} = \text{constant}\,.} \tag{11}$$