AST 376 Cosmology — Lecture Notes

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COSMIC SPACETIME

A coordinate system in general relativity reflects a choice of observers. If we are smart we can choose reference frames that simplify the equations and lend themselves to the underlying physics involved. Overall, our goal is to find the "metric" of the Universe!

The metric of the Universe

Recall the general spacetime interval:

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (\text{"GR Line Element"})\,,} \tag{1}$$

where the true physics *is* the left hand side, i.e. the proper length *is* ds and the proper time *is* $d\tau = \sqrt{-ds^2/c^2}$. However, we can only access the LHS through the metric on the RHS. This is important because now we are asking for the form of the metric in a given coordinate system. For example, the coordinates we used in special relativity are $dx^\mu = (cdt, dx, dy, dz)$. Therefore, we cover spacetime with a coordinate system (or many patches) and solve the Einstein equation,

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (\text{"Einstein Equation"})\,,} \tag{2}$$

for the ten independent components of the "metric tensor"

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ & g_{11} & g_{12} & g_{13} \\ & & g_{22} & g_{23} \\ & & & g_{33} \end{pmatrix} . \tag{3}$$

We only show the upper portion of $g_{\mu\nu}$ because symmetry requires that $g_{\mu\nu} = g_{\nu\mu}$. This turns out to be too difficult! Only supercomputers can solve Eq. 2 in full generality so we need a shortcut.

Trick: We must find an observer for whom the Universe looks "as simple as possible." Specifically, our **fundamental observers** are chosen so that they move with the Hubble flow, i.e. we ignore peculiar motions. Furthermore, we define "cosmic time" as the proper time along the worldline of these inertial ("freely-falling") observers. With this choice we are guaranteed to have the time coordinate t be "orthogonal" to the remaining three spatial coordinates so that

$$ds^2 = -c^2 dt^2 + d\ell^2\,, \qquad \text{i.e.} \quad g_{00} = -1 \quad \text{and} \quad g_{0i} = 0 \quad \text{for} \quad i = \{1,2,3\}\,. \tag{4}$$

Q: What about the spatial part $d\ell^2$?
A: We have some very specific clues about this from empirical observations of nature:

(i) The Universe is **flat** (Euclidean) so the Pythagorean Theorem holds.

(ii) The Universe undergoes **cosmic expansion** as described by the scale factor $a = a(t)$.

(iii) The Universe is **isotropic** so we may use spherical coordinates with "us" at the center.

Recall: The scale factor is normalized such that the present day value is one, i.e. $a(t_0) = a_0 \equiv 1$. This provides the connection between physical $\mathbf{r}$ and comoving $\mathbf{x}$ coords. according to $\mathbf{r} = a(t)\mathbf{x}$.

From these points we are led to the form of the spatial part of the metric in Eq. 4:

$$\boxed{d\ell^2 = a^2\left(dx^2 + dy^2 + dz^2\right) = a^2\left(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2\right)\ .} \tag{5}$$

Here we have used the three-dimensional version of Pythagorean's Theorem, multiplied by the time-dependent scale factor, and made a transformation to spherical coordinates. The variables (r, ϑ, φ) are called "comoving coordinates" because they are the unique labels (telephone numbers) that correspond to points in space which remain constant with cosmic expansion. When we combine Eqs. 4 and 5 we arrive at one of the most important equations in all of cosmology

$$\boxed{ds^2 = -c^2 dt^2 + a^2\left(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2\right) \qquad (\text{"Robertson-Walker (RW) metric"})\,.} \tag{6}$$

Memorize this and never forget it the rest of your career!
Note: This has different names in various circles, e.g. Friedmann-Robertson-Walker (FRW) metric.
Thus, from symmetry arguments we have figured out the metric of the Universe:

$$\boxed{g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2\vartheta \end{pmatrix} \qquad \text{for (spherical) comoving coordinates } (r, \vartheta, \varphi)\,.} \tag{7}$$

Aside: What is the biggest indication that we live in a flat universe?

The first peak of the CMB: At ~ 0.5 Myr the Universe was ~ 1000 times smaller than its current size. Gas and radiation were tightly coupled and could easily affect each other. At various locations in space small perturbations would make ripples which propagate at the speed of sound, i.e. of pressure waves. The sound horizon at the CMB leaves a measurable imprint – a characteristic scale between hot and cold spots. The geometry of space determines how large these distances appear on the angular backdrop of the sky. A closed universe has a larger angle while an open universe has a smaller one. However, the measured angle from CMB anisotropy experiments tells us the Universe is flat!