AST 376 Cosmology — Lecture Notes

Prof. Volker Bromm — TA: Aaron Smith

(Dated: February 11, 2014)

GENERAL RELATIVITY (GR) – A VERY BRIEF INTRODUCTION (CONT.)

We pick up the story with the goal of arriving at the general relativistic field equation.

Einstein Field Equation

We have an ansatz for the field equation based on a generalization of Poisson's Equation:

$$\nabla^2 \varphi = 4\pi G \rho \qquad \dashrightarrow \qquad -\nabla^2 g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \, .$$

The LHS is still not correct, but this is how far Einstein was around 1910. The mathematical framework behind relativity had already been developed 50 years earlier but physicists didn't know about it. The theory is quite elegant but also very difficult. Even when Einstein had a mathematician friend help him out, the new tools took some time to learn before he could finish his theory.

The main problem with the LHS is that ∇^2 is not a tensor operation. Briefly, a tensor is a sort of generalized vector so it is independent of the coordinate system used. This is useful because the coordinates are not physical, they only interpret the physics. Therefore, we can divorce ourselves from the coordinates and write laws such as $\vec{F} = m\vec{a}$ which are valid for all frames.

Off the record: There are also specific requirements that tensors must follow for this to be true. Formally, a tensor transforms in a specific manner under a change of coordinates from $x^\mu = (x^0, x^1, x^2, x^3)$ to $x^{\mu'} = (x^{0'}, x^{1'}, x^{2'}, x^{3'})$

$$p^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} p^\mu$$

for a vector and the stress energy tensor

$$T_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} T_{\mu\nu}$$

provides an example for higher rank tensors.

Additional reasons ∇^2 is not what we want in relativity are *(i)* it only acts spatially, i.e. space and time should be on equal footing, and *(ii)* both sides need to be divergence-less, i.e. $\mathrm{Div}(G_{\mu\nu}) = 0$. The only combination obeying these properties is the so-called **Einstein tensor** $G_{\mu\nu}$:

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (\text{"Relativistic field equation"})\, .} \tag{1}$$

Off the record: Here the Ricci tensor is built up from the Christoffel symbols:

$$R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} \qquad (\text{“Ricci tensor”})\,, \tag{2}$$

and the Ricci scalar R is the contraction of the Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu} \qquad (\text{“Ricci scalar”})\,. \tag{3}$$

The Christoffel symbols themselves are

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right) \qquad (\text{“Christoffel symbols”})\,. \tag{4}$$

Note: If we adopt the notation of using commas to denote derivatives ($f_{,\alpha} \equiv df/dx^{\alpha}$) then Eq. 4 simplifies to an expression that is much easier to remember:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})\,.$$

Schematically Eq. 1 can be understood as

$$\begin{pmatrix}\text{curvature}\\ \text{of spacetime}\end{pmatrix} = \frac{8\pi G}{c^4}\begin{pmatrix}\text{energy}\\ \text{density}\end{pmatrix}\,. \tag{5}$$

Equation 5 relates the fact that ‘matter tells space(time) how to curve.’ In other words, the **‘Einstein Machine’** tells us that if we have a *source of gravity* we can figure out the *geometry of spacetime*! This is important because the geometry then ‘tells matter how to move.’

Note: The Einstein field equation represents 10 independent, but coupled, equations instead of 16 because the tensors are symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$.

Relativistic Equation of Motion

Rule: Freely falling particles travel on **geodesics** or ‘straightest’ paths through spacetime. To find this “world line” we use the Calculus of Variations to minimize the spacetime interval:

$$\boxed{\int_A^B ds = \text{Extremal} \qquad \Longleftrightarrow \qquad \delta\int_A^B ds = 0 \qquad (\text{“Calculus of Variations”})\,.} \tag{6}$$

If sources of mass/energy stretch spacetime then we need an equation of motion that understands spacetime. This is where geodesics come in! **Note:** The geodesic equation is $\ddot{x}^{\alpha} = -\Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$.

A Special Case: Motion of Photons

Consider photons moving in SR with $dy = dz = 0$. The universality of the speed of light requires

$$c = \frac{dx}{dt} \quad \Rightarrow \quad ds^2 = -c^2dt^2 + dx^2 = 0\,,$$

so that “photons travel along **null geodesics**.” This is also true in GR because of our freedom to choose a locally freely-falling frame ($ds \stackrel{*}{=} 0$). We may patch an arbitrary number of local frames together to show $ds^2 = 0$ for photons in general.