AST 376 Cosmology — Lecture Notes

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GENERAL RELATIVITY (GR) – A VERY BRIEF INTRODUCTION (CONT.)

In-class Review

We don't have time to go into GR fully! If you want a pedagogical resource here is an ebook available through the UT library system.

Recall from our previous discussion that coordinates can be anything. Some examples we have used are $dx^0 = cdt$, $dx^1 = dx$, etc. and the polar coordinates (r, φ). Why do we use superscripts and subscripts when we wrote $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$? Mathematically, there is a deep meaning distinguishing contravariant and covariant vectors/tensors but for our purposes it is to remind us about the summation rule – repeated indices always come in up/down pairs.

So far we have the following descriptions of gravity:

$$\boxed{\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\vec{\nabla}\phi \qquad \text{(Equation of Motion)}}$$

and

$$\boxed{\nabla^2\varphi = 4\pi G\rho \qquad \text{(Poisson's Equation)}\,.}$$

We need one more connection to provide a full transition to general relativity.

Gravitational Redshift

We will now use a pseudo-Newtonian argument to explain the gravitational redshift of light moving away from a massive object such as a planet or star. Originally the emitted energy of the photon at the surface R is $\epsilon_{\rm em} = h\nu_{\rm em}$ but far away from the star we find a different energy given by $\epsilon_\infty = h\nu_\infty$. The photon losses energy, which comes from the work required to climb out of the potential well of the star. If we define the 'effective mass' of the photon as

$$m_{\rm eff} \equiv \frac{\epsilon_{\rm em}}{c^2} \qquad \text{where} \qquad \epsilon_{\rm em} = h\nu_{\rm em} = \frac{hc}{\lambda_{\rm em}}$$

then the work done is

$$W = -\int_R^\infty \vec{F}\cdot d\vec{r} = \frac{GM}{R} m_{\rm eff} = -\varphi(R)\frac{h\nu_{\rm em}}{c^2}\,.$$

Therefore, conservation of energy requires the energy to be lowered by a small amount:

$$h\nu_\infty = h\nu_{\rm em} - W = h\nu_{\rm em} + \varphi\frac{h\nu_{\rm em}}{c^2} = h\nu_{\rm em}\left(1 + \frac{\varphi}{c^2}\right)\,,$$

or simply

$$\boxed{\nu_\infty = \nu_{\rm em}\left(1 + \frac{\varphi}{c^2}\right) \qquad \text{(Pseudo-Newtonian Gravitational Redshift)}\,.} \qquad (1)$$

This can be expressed as a dimensionless redshift as

$$\frac{\Delta\nu}{\nu_{\rm em}} = \frac{\nu_\infty - \nu_{\rm em}}{\nu_{\rm em}} = \frac{|\varphi|}{c^2}\,.$$

Newton gets the right answer for the wrong reasons! This gives us an order of magnitude estimation for the strength of the gravity. Indeed, it roughly corresponds to $R_{\rm S}/R$.

Q: Why is the gravitational redshift important?
A: Gravity impacts the flow of time! Recall that $\Delta t \propto \nu^{-1}$ so using light clocks in Einstein's thought experiments we can think of a gravitational time delay:

$$\boxed{\Delta t(R) = \Delta t_\infty \left(1 + \frac{\varphi}{c^2}\right) \qquad \text{(Pseudo-Newtonian Gravitational Time Delay)}\,.} \tag{2}$$

Now we are changing time intervals similar to what was done previously with the metric.

Consider an observer in the potential well at $r = R$. Their clock measures the proper time so

$$\Delta\tau = \Delta t(R) \qquad \text{and} \qquad \Delta t_\infty = \Delta t\,.$$

Plugging this into Equation 2, taking the square of both sides, and expanding in the (Newtonian) weak field limit, i.e. Taylor expanding, gives

$$\Delta\tau^2 = \Delta t^2 \left(1 + \frac{\varphi}{c^2}\right)^2 \to \Delta t^2 \left(1 + \frac{2\varphi}{c^2}\right)\,.$$

With this we may reassemble the entire spacetime metric of Newtonian physics:

$$\boxed{ds^2 = -c^2 \left(1 + \frac{2\varphi}{c^2}\right) dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{(``Weak Field Metric'')}\,.} \tag{3}$$

This is the geometric version of Newtonian gravity! In general the metric may be much more complicated so we introduce the general metric $g_{\mu\nu}$ so the line element becomes

$$ds^2 = g_{\mu\nu} dx^\mu dx \nu\,,$$

and decoding Equation 3 gives the first entry

$$\boxed{g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right)\,.} \tag{4}$$

The 'real' definition of gravity is curvature of spacetime as encoded by second derivatives of $g_{\mu\nu}$:

$$-\nabla^2 g_{00} = \frac{2}{c^2}\nabla^2\varphi = \frac{8\pi G}{c^2}\rho\,.$$

There are a number of problems with this heuristic equation. First of all in GR all forms of energy are sources of gravity:

Newton: ρ – mass density only

Einstein: ρc^2 – mass-energy density **AND** P – pressure in all 3 spatial directions.

We can neatly organize these contributions by introducing the stress-energy tensor $T_{\mu\nu}$. One can choose a special coordinate system representing the fluid rest frame where there are no off-diagonal components:

$$\boxed{T_{\mu\nu} = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad \text{(“Stress-energy tensor”).}} \tag{5}$$

Why are the three pressures? They are independent degrees of freedom in the isotropic case. By taking the trace of this matrix $T = T^{\mu}_{\mu}$ and dividing by c^2 we get the “average” or effective density:

$$\boxed{\rho_{\text{eff}} = \rho + \frac{3P}{c^2}\,.} \tag{6}$$

This is why massive stars ($M > 40\ \text{M}_{\odot}$) inevitably collapse into black holes. At some point the standard trick of balancing gravity with the interior pressure does not work because pressure itself becomes an additional source of gravity!